MATHEMATICAL METHODS

SOLUTION OF LINEAR SYSTEMS

I YEAR B.Tech

By

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MATRICES

**Matrix:** The arrangement of set of elements in the form of rows and columns is called as Matrix. The elements of the matrix being Real (or) Complex Numbers.

**Order of the Matrix:** The number of rows and columns represents the order of the matrix. It is denoted by \( m \times n \), where \( m \) is number of rows and \( n \) is number of columns.

**Ex:** \( A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \) is \( 2 \times 3 \) matrix.

**Note:** Matrix is a system of representation and it does not have any Numerical value.

**Types of Matrices**

- **Rectangular Matrix:** A matrix is said to be rectangular, if the number of rows and number of columns are not equal.

  **Ex:** \( A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \) is a rectangular matrix.

- **Square Matrix:** A matrix is said to be square, if the number of rows and number of columns are equal.

  **Ex:** \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \) is a Square matrix.

- **Row Matrix:** A matrix is said to be row matrix, if it contains only one row.

  **Ex:** \( A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \) is a row matrix.

- **Column Matrix:** A matrix is said to be column matrix, if it contains only one column.

  **Ex:** \( A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) is a column matrix.

- **Diagonal Matrix:** A square matrix \( A_{n \times n} \) is said to be diagonal matrix if \( a_{ij} = 0 \ \forall \ i \neq j \)

  (Or)

  A Square matrix is said to be diagonal matrix, if all the elements except principle diagonal elements are zeros.

  - The elements on the diagonal are known as principle diagonal elements.

  - The diagonal matrix is represented by \( A = diag[a_{11} \ a_{22} \ ... \ a_{nn}] \)

  **Ex:** If \( A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \) then \( A = diag[1 \ 2] \)

- **Trace of a Matrix:** Suppose \( A \) is a square matrix, then the trace of \( A \) is defined as the sum of its diagonal elements.

  \[ i.e. \ Tr(A) = a_{11} + a_{22} + \cdots + a_{nn} \]

  - \( Tr(A + B) = Tr(A) + Tr(B) \)
Scalar Matrix: A Square matrix $A_{n \times n}$ is said to be a Scalar matrix if

$$a_{ij} = 0 \forall i \neq j$$

$$a_{ij} = k \forall i = j$$

(Or)

A diagonal matrix is said to be a Scalar matrix, if all the elements of the principle diagonal are equal.

i.e. $a_{ij} = k \forall i = j$

- Trace of a Scalar matrix is $nk$.

Unit Matrix (or) Identity Matrix: A Square matrix $A_{n \times n}$ is said to be a Unit (or) Identity matrix if

$$a_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

(Or)

A Scalar matrix is said to be a Unit matrix if the scalar $k = 1$

Ex: $I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- Unit matrix is denoted by $I$.

- The Trace of a Unit Matrix is $n$, where order of the matrix is $n \times n$.

Transpose of a Matrix: Suppose $A$ is a $m \times n$ matrix, then transpose of $A$ is denoted by $A'$ (or) $A^T$ and is obtained by interchanging of rows and columns of $A$.

Ex: If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -5 & 4 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 0 \\ -2 & -5 \\ 3 & 4 \end{bmatrix}$

- If $A$ is of Order $m \times n$, then $A^T$ is of Order $n \times m$

- If $A$ is a square matrix, then $Tr(A) = Tr(A^T)$

- $(A + B)^T = A^T + B^T$

- $(AB)^T = B^T A^T$

- $(kA)^T = kA^T$

- If $A$ is a scalar matrix then $A^T = A$

- $(A^T)^T = A$

- $I^T = I$

Upper Triangular Matrix: A matrix $A_{m \times n}$ is said to be an Upper Triangular matrix, if

$$a_{ij} = 0 \forall i > j.$$ 

Ex: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 7 & 2 & 1 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is Upper Triangular matrix

- In a square matrix, if all the elements below the principle diagonal are zero, then it is an Upper Triangular Matrix

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 7 \end{bmatrix}$ is a Upper Triangular matrix.

Lower Triangular Matrix: A matrix $A_{m \times n}$ is said to be a Lower Triangular matrix, if

$$a_{ij} = 0 \forall i < j.$$
Ex: \( A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 9 & 8 & 0 \end{bmatrix} \) is Upper Triangular matrix

- In a square matrix, if all the elements above the principal diagonal are zero, then it is an Lower Triangular Matrix.

Ex: \( \begin{bmatrix} 1 & 0 \\ 2 & 4 & 0 \\ 3 & 7 & 7 \end{bmatrix} \) is a Lower Triangular matrix.

- Diagonal matrix is Lower as well as Upper Triangular matrix.

**Equality of two matrix:** Two matrices \( A_{m \times n}, B_{m \times n} \) are said to be equal if \( a_{ij} = b_{ij} \; \forall \; i, j \)

**Properties on Addition and Multiplication of Matrices**

- Addition of Matrices is Associative and Commutative
- Matrix multiplication is Associative
- Matrix multiplication need not be Commutative
- Matrix multiplication is distributive over addition
  
i.e. \( A(B + C) = AB + AC \) (Left Distributive Law)
  
\[(B + C)A = BA + CA \) (Right Distributive Law)
- Matrix multiplication is possible only if the number of columns of first matrix is equal to the number of rows of second matrix.

**Symmetric Matrix:** A Square matrix \( A_{n \times n} \) is said to be symmetric matrix if \( A^T = A \)

i.e. \( a_{ij} = a_{ji} \; \forall \; i, j \)

- Identity matrix is a symmetric matrix.
- Zero square matrix is symmetric i.e. \( O_{n \times n} \).

- Number of Independent elements in a symmetric matrix are \( \frac{n(n+1)}{2} \), \( n \) is order.

**Skew Symmetric Matrix:** A Square matrix \( A_{n \times n} \) is said to be symmetric matrix if \( A^T = -A \)

i.e. \( a_{ij} = -a_{ji} \; \forall \; i, j \)

It is denoted by \( A' \)

- Zero square matrix is symmetric i.e. \( O_{n \times n} \).
- The elements on the principle diagonal are zero.
- Number of Independent elements in a skew symmetric matrix are \( \frac{n(n-1)}{2} \), \( n \) is order.

Ex: 1) \( A = \begin{bmatrix} 1 & 3 \\ -3 & 0 \end{bmatrix} \) is not a skew symmetric matrix

2) \( A = \begin{bmatrix} 0 & -3 & -5 \\ 3 & 0 & -9 \\ 5 & 9 & 0 \end{bmatrix} \) is a skew symmetric matrix.

**Theorem**

Every Square matrix can be expressed as the sum of a symmetric and skew-symmetric matrices.
Sol: Let us consider \( A \) to be any matrix.

Now, \( A = \frac{1}{2} (2A) \)

\[ = \frac{1}{2} (A + A) \]

\[ = \frac{1}{2} [(A + A^T) + (A - A^T)] \]

\[ = \frac{1}{2} (A + A^T) + \frac{1}{2} (A - A^T) \]

This is in the form of \( A = B + C \), where \( B = \frac{1}{2} (A + A^T), C = \frac{1}{2} (A - A^T) \)

Now, we shall prove that one is symmetric and other one is skew symmetric.

Let us consider \( B = \frac{1}{2} (A + A^T) \)

\[ \Rightarrow B^T = \left[ \frac{1}{2} (A + A^T) \right]^T \]

\[ = \frac{1}{2} (A + A^T)^T = \frac{1}{2} [A^T + (A^T)^T] \]

\[ \therefore (A + B)^T = A^T + B^T \]

\[ \Rightarrow \frac{1}{2} [A^T + A] = B \]

\[ \Rightarrow B^T = B \]

\( \therefore B \) is Symmetric Matrix

Again, let us consider \( C = \frac{1}{2} (A - A^T) \)

\[ \Rightarrow C^T = \left[ \frac{1}{2} (A - A^T) \right]^T \]

\[ = \frac{1}{2} (A - A^T)^T = \frac{1}{2} [A^T - (A^T)^T] \]

\[ \therefore (A - B)^T = A^T - B^T \]

\[ \Rightarrow \frac{1}{2} [A^T - A] = -\frac{1}{2} (A - A^T) = -C \]

\[ \Rightarrow C^T = -C \]

\( \therefore C \) is Skew-Symmetric Matrix

Hence, every square matrix can be expressed as sum of symmetric and skew-symmetric matrices.

**Conjugate Matrix:** Suppose \( A \) is any matrix, then the conjugate of the matrix \( A \) is denoted by \( \overline{A} \) and is defined as the matrix obtained by taking the conjugate of every element of \( A \).

- Conjugate of \( a + ib \) is \( a - ib \)
- \( \overline{\overline{A}} = A \)
- \( \overline{A \cdot B} = \overline{A} \cdot \overline{B} \)
- \( \overline{A + B} = \overline{A} + \overline{B} \)

**Ex:** If \( A = \begin{bmatrix} 1 & 2 + 3i \\ 3 - 4i & -2i \end{bmatrix} \Rightarrow \overline{A} = \begin{bmatrix} 1 & 2 - 3i \\ 3 + 4i & 2i \end{bmatrix} \)

**Conjugate Transpose of a matrix (or) Transpose conjugate of a matrix:** Suppose \( A \) is any square matrix, then the transpose of the conjugate of \( A \) is called Transpose conjugate of \( A \).

It is denoted by \( A^\theta = (\overline{A})^T = (\overline{A^T}) \).

**Ex:** If \( A = \begin{bmatrix} 1 - i & -2i \\ 4 - 3i & 5 - 4i \end{bmatrix} \) then \( \overline{A} = \begin{bmatrix} 1 + i & 2i \\ 4 + 3i & 5 + 4i \end{bmatrix} \)

Now, \( (\overline{A})^T = \begin{bmatrix} 1 + i & 4 + 3i \\ 2i & 5 + 4i \end{bmatrix} = A^\theta \)
Orthogonal Matrix: A square matrix $A$ is said to be Orthogonal if $A A^T = A^T A = I$

Ex: $A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$

- If $A$ is orthogonal, then $A^T$ is also orthogonal.
- If $A, B$ are orthogonal matrices, then $AB$ is orthogonal.

Elementary Row Operations on a Matrix

There are three elementary row operations on a matrix. They are

- Interchange of any two Rows.
- Multiplication of the elements of any row with a non-zero scalar (or constant)
- Multiplication of elements of a row with a scalar and added to the corresponding elements of other row.

Note: If these operations are applied to columns of a matrix, then it is referred as elementary column operation on a matrix.

Elementary Matrix: A matrix which is obtained by the application of any one of the elementary operation on Identity matrix (or) Unit matrix is called as Elementary Matrix

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a Elementary matrix. (∵ $R_1 \leftrightarrow R_2$)

- To perform any elementary Row operations on a matrix $A$, pre multiply $A$ with corresponding elementary matrix.
- To perform any elementary column operation on a matrix $A$, post multiply $A$ with corresponding elementary matrix.

Determinant of a Matrix

Determinant of a Matrix: For every square matrix, we associate a scalar called determinant of the matrix.

- If $A$ is any matrix, then the determinant of a Matrix is denoted by $|A|$
- The determinant of a matrix is a function, where the domain set is the set of all square matrices and Image set is the set of scalars.

- **Determinant of $1 \times 1$ matrix:** If $A = [a]_{1 \times 1}$ matrix then $|A| = a$
- **Determinant of $2 \times 2$ matrix:** If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $|A| = ad - bc$
- **Determinant of $3 \times 3$ matrix:** If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then $|A| = a \begin{vmatrix} e & f \\ i & a \end{vmatrix} - b \begin{vmatrix} d & f \\ g & a \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$
Minor of an Element: Let $A = (a_{ij})_{n \times n}$ be a matrix, then minor of an element $a_{ij}$ is denoted by $M_{ij}$ and is defined as the determinant of the sub-matrix obtained by omitting $i^{th}$ row and $j^{th}$ column of the matrix.

Cofactor of an element: Let $A = (a_{ij})_{n \times n}$ be a matrix, then cofactor of an element $a_{ij}$ is denoted by $A_{ij}$ and is defined as $A_{ij} = (-1)^{i+j} M_{ij}$

Cofactor Matrix: If we find the cofactor of an element for every element in the matrix, then the resultant matrix is called as Cofactor Matrix.

Determinant of a $n \times n$ matrix: Let $A = (a_{ij})_{n \times n}$ be a matrix, then the determinant of the matrix is defined as the sum of the product of elements of $i^{th}$ row (or) $j^{th}$ column with corresponding cofactors and is given by

$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in}$ (For $i^{th}$ row)

$|A| = |A^T|$

If any two rows (or) columns are interchanged, then the determinant of resulting matrix is $-|A|$.

If any row (or) column is zero then $|A| = 0$.

If any row (or) column is a scalar multiple of other row (or) column, then $|A| = 0$.

If any two rows (or) columns are identical then $|A| = 0$.

If any row (or) column of $A$ is multiplied with a non-zero scalar $\lambda$, then determinant if resulting matrix is $\lambda \  |A|$.  

If $A_{n \times n}$ is multiplied with a non-zero scalar $\lambda$, then determinant of the resulting matrix is given by $\lambda^n \  |A|$.  

Determinant of the diagonal matrix is product of diagonal elements.

Determinant of the Triangular matrix (Upper or Lower) = product of the diagonal elements.

$|AB| = |A||B|$

If any row (or) column is the sum of two elements type, then determinant of a matrix is equal to the sum of the determinants of matrices obtained by separating the row (or) column.

Ex:

\[
\begin{vmatrix}
 a & b & c + d \\
 p & q & r + s \\
 w & x & y + z
\end{vmatrix} = \begin{vmatrix}
 a & b & c \\
 p & q & r \\
 w & x & y
\end{vmatrix} + \begin{vmatrix}
 a & b & d \\
 p & q & r \\
 w & x & z
\end{vmatrix}
\]

Adjoint Matrix: Suppose $A$ is a square matrix of $n \times n$ order, then adjoint of $A$ is denoted by $adjA$ and is defined as the Transpose of the cofactor matrix of $A$.

Ex: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $adjA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$A(adjA) = (adjA)A = |A|I$
i.e. Every square matrix $A$ and its adjoint matrix are commutative w.r.t multiplication.

- $|adjA| = |A|^{n-1}$, $|A| \neq 0$
- $|A adjA| = |A|^n$
- $adj(AB) = (adjB)(adjA)$
- If $A$ is a $3 \times 3$ scalar matrix with scalar $k$, then $adj(A) = k^2I$.

**Singular Matrix:** A square matrix $A$ is said to be singular if $|A| = 0$.

**Non-singular Matrix:** A square matrix $A$ is said to be non-singular if $|A| \neq 0$.

**Inverse of a Matrix:** A square matrix $A_{n \times n}$ is said to be invertible if there exists a matrix $B_{n \times n}$ such that $AB = BA = I$, where $B$ is called inverse of $A$.

- Necessary and sufficient condition for a square matrix $A$ to be invertible is that $|A| \neq 0$.
- If $A, B$ are two invertible matrices, then $AB$ is also invertible.
- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- If $k \neq 0$ is a scalar, $A$ is an invertible matrix, then $(kA)^{-1} = k^{-1}A^{-1} = \frac{1}{k}A^{-1}$
- Addition of two invertible matrices need not be invertible.
- If $A, B$ are two non-zero matrices such that $AB = 0$, then $A, B$ are singular.
- If $A$ is orthogonal matrix, then inverse of $A$ is $A^T$. ($\therefore AA^T = A^TA = I$)
- If $A$ is Unitary matrix, then $A^\theta$ is inverse of $A$. ($\therefore AA^\theta = A^\theta A = I$)
- $(A^T)^{-1} = (A^{-1})^T$
- $(A^\theta)^{-1} = (A^{-1})^\theta$
- Inverse of an identity matrix is identity itself.
- If $A$ is a non-singular matrix, then $A^{-1} = \frac{adjA}{|A|}$
- If $A$ is a non-singular matrix, then $AB = AC \Rightarrow B = C$
- If $AB = I$ then $BA = I$

**Procedure to find Inverse of a Matrix**

In order to find the determinant of a $3 \times 3$ matrix, we have to follow the procedure given below.

Let us consider the given matrix to be $A$

**Step 1:** Find determinant of $A$ i.e. if $|A| \neq 0$ then only inverse exists. Otherwise not (i.e. $|A| = 0$)

**Step 2:** Find Minor of each element in the matrix $A$.

**Step 3:** Find the Co-factor matrix.

**Step 4:** Transpose of the co-factor matrix, which is known as $adjA$

**Step 5:** Inverse of $A$: $A^{-1} = \frac{adjA}{|A|}$

**Calculation of Inverse using Row operations**
Procedure: If \( A \) is a \( n \times n \) square matrix such that \( |A| \neq 0 \), then calculation of Inverse using Row operation is as follows:

- Consider a matrix \([A / I]\) and now convert \( A \) to \( I \) using row operations. Finally we get a matrix of the form \([I / B]\), where \( B \) is called as Inverse of \( A \).

**Row reduced Echelon Form of matrix**

Suppose \( A \) is a \( n \times n \) matrix, then it is said to be in row reduced to echelon form, if it satisfies the following conditions.

- The number of zeros before the first non-zero element of any row is greater than the number of zeros before the first non-zero element of preceding (next) row.
- All the zero rows, if any, are represented after the non-zero rows.
  - Zero matrix and Identity matrix are always in Echelon form.
  - Row reduced echelon form is similar to the upper triangular matrix.
  - In echelon form, the number of non-zero rows represents the Independent rows of a matrix.
  - The number of non-zero rows in an echelon form represents Rank of the matrix.

**Theorem**

Prove that the Inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal.

**Proof:** Let us consider \( A \) to be the square matrix.

Now, given that \( A \) is Orthogonal \( \Rightarrow A A^T = A^T A = I \)

Now, we have to prove “Inverse of an orthogonal matrix is orthogonal”

For that, consider \( A A^T = I \)

\[
A^{-T} (A^T)^{-T} = I
\]

\[
A^{-T} (A^{-})^T = I
\]

\[
A^{-T} \text{ is Orthogonal}
\]

Now, let us prove transpose of an orthogonal matrix is orthogonal

Given that \( A \) is Orthogonal \( \Rightarrow A A^T = A^T A = I \)

Consider \( A A^T = I \)

Now, \((A A^T)^T = I^T\)

\[
(A^T)^T A^T = I
\]

\[
A^T \text{ is orthogonal.}
\]

FOR CONFIRMATION

If \( A \) is Orthogonal \( \Rightarrow A A^T = A^T A = I \)

If \( A^{-T} \) is Orthogonal \( \Rightarrow A^{-T} (A^{-})^T = (A^{-})^T A^{-T} = I \)

FOR CONFIRMATION

If \( A \) is Orthogonal \( \Rightarrow A A^T = A^T A = I \)

If \( A^T \) is Orthogonal \( \Rightarrow A^T (A^T)^T = (A^T)^T A^T = I \)
Rank of the Matrix

If $A$ is a non-zero matrix, then $A$ is said to be the matrix of rank $r$, if

i. $A$ has atleast one non-zero minor of order $r$, and

ii. Every $(r + 1)^{th}$ order minor of $A$ vanishes.

The order of the largest non-zero minor of a matrix $A$ is called Rank of the matrix. It is denoted by $\rho(A)$.

- When $A = 0$, then $\rho(A) = 0$.
- Rank of $I = n$, $n$ is order of the matrix.
- If $|A| \neq 0$ for $A_{n \times n}$ matrix, then $\rho(A) = n$.
- For $A_{n \times n}$ matrix, $\rho(A) \leq n$.
- If $\rho(A) = r$, then the determinant of a sub-matrix, where order $> r$ is equal to zero.
- The minimum value of a Rank for a non-zero matrix is one.

- $\rho(AB) \leq \rho(A) \& \rho(AB) \leq \rho(B)$
- $\rho(A + B) \leq \rho(A) + \rho(B)$
- $\rho(A - B) \geq \rho(A) - \rho(B)$

Problem

Find the rank of the following matrix

$$
\begin{bmatrix}
2 & 1 & 3 & 5 \\
4 & 2 & 1 & 3 \\
8 & 4 & 7 & 13 \\
8 & 4 & -3 & -1
\end{bmatrix}
$$

Sol: Let us consider $A =
\begin{bmatrix}
2 & 1 & 3 & 5 \\
4 & 2 & 1 & 3 \\
8 & 4 & 7 & 13 \\
8 & 4 & -3 & -1
\end{bmatrix}$

$R_2 \rightarrow R_2 - 2R_1$
$R_3 \rightarrow R_2 - 4R_1$
$R_4 \rightarrow R_4 - 4R_1$

Therefore, the number of non-zero rows in the row echelon form of the matrix is 2. Hence rank of the matrix is 2.

Problem: Reduce the matrix

$$
\begin{bmatrix}
1 & 2 & 3 & 0 \\
2 & 4 & 3 & 2 \\
3 & 2 & 1 & 3 \\
6 & 8 & 7 & 5
\end{bmatrix}
$$

into echelon form and hence find its rank.

Sol: Let us consider given matrix to be $A$
Now, this is in Echelon form and the number of non-zero rows is 3

Hence, \( \rho(A) = 3 \)

Equallence of two matrices

Suppose \( A \) and \( B \) are two matrices, then \( B \) is said to be row equivalent to \( A \), if it is obtained by applying finite number of row operations on \( A \). It is denoted by \( B \overset{R}{\sim} A \).

Similarly, \( B \) is said to be column equivalent to \( A \), if it is obtained by applying finite number of column operations on \( A \). It is denoted by \( B \overset{C}{\sim} A \).

- For equivalent matrices Rank does not Alter (i.e. does not change)
- Equallence of matrices is an Equallence relation
- Here Equallence \( \implies \) following three laws should satisfy
  - Reflexive: \( A \overset{R}{\sim} A \)
  - Symmetric: \( A \overset{R}{\sim} B \implies B \overset{R}{\sim} A \)
  - Transitive: \( A \overset{R}{\sim} B, B \overset{R}{\sim} C \implies A \overset{R}{\sim} C \)

Normal Form of a Matrix

Suppose \( A \) is any matrix, then we can convert \( A \) into any one of the following forms

\[
\begin{bmatrix}
I_r & 0 \\
0 & 0
\end{bmatrix}
\text{ (Or) }
\begin{bmatrix}
I_r \\
0
\end{bmatrix}
\text{ (Or) }
\begin{bmatrix}
I_r & 0
\end{bmatrix}
\]
These forms are called as Normal forms of the matrix $A$. (Or canonical forms)

**Procedure to find Normal form of the matrix $A$.**

**Aim:** Define two non-singular matrices $P$ & $Q$ such that $PAQ$ is in Normal Form.

**Step 1:** Let us consider $A$ is the given matrix of order $m \times n$.

**Step 2:** Rewrite $A$ as $A = I_mA \ I_n$

**Step 3:** Reduce the matrix $A$ (L.H.S) in to canonical form using elementary operations provided every row operation which is applied on $A$ (L.H.S), should be performed on pre-factor $I_m$(R.H.S). And every column operation which is applied on $A$ (L.H.S), should be performed on post-factor $I_n$ (R.H.S).

**Step 4:** Continue this process until the matrix $A$ at L.H.S takes the normal form.

**Step 5:** Finally, we get $I_r = PAQ$, $r$ is rank of the matrix $A$.

- The order of Identity sub-matrix of the Normal form of $A$ represents Rank of the matrix of $A$.
- Every matrix can be converted into Normal form using finite number of row and column operations.
- If we convert the matrix $A$ in to Normal form then $\exists$ two non-singular matrices $P$ and $Q$ such that $PAQ = Normal \ Form$, where $P$ and $Q$ are the product of elementary matrices.
- Every Elementary matrix is a non-singular matrix.

**SYSTEM OF LINEAR EQUATIONS**

The system of Linear equations is of two types.

- Non-Homogeneous System of Linear Equations
- Homogeneous System of Linear Equations.

**Non-Homogeneous System of Linear Equations**

The system of equations which are in the form

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
then, the above system of equations is known as Non-Homogeneous system of Linear equations and it is represented in the matrix form as follows:

The above system of equation can be represented in the form of $AX = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

**Solution of $AX = B$**

The set of values $\{x_1, x_2, \ldots, x_n\}$ is said to be a solution to $AX = B$ if it satisfies all the equations.

**Consistent system of equations**

A system of equations $AX = B$ is said to be consistent if it has a solution. Otherwise, it is called as Inconsistent (i.e. no solution).

**Augmented Matrix**

The matrix $[A / B]$ is called as an Augmented matrix.

**Necessary and Sufficient condition** for $AX = B$ to be consistent is that $\rho[A/B] = \rho[A]$.

- If $\rho[A/B] = \rho[A] = r = n$ (number of variables (or) unknowns), then $AX = B$ has unique solution.
- If $m = n$ (i.e. Number of equations = Number of unknowns) and $|A| \neq 0$, then $AX = B$ has Uniquely solution
- If $\rho[A/B] = \rho[A] = r < n$ (unknowns) and $|A| \neq 0$, then $AX = B$ has Infinitely many Solutions.
- If $m = n$ (i.e. Number of equations = Number of unknowns) and $|A| = 0$, then $AX = B$ has Infinitely many solutions.
- If $m > n$ (i.e. Number of equations > Number of unknowns), then $AX = B$ has Infinitely many solutions if $\rho[A/B] = \rho[A] = r < n$.

**Procedure for solving $AX = B$**

Let $AX = B$ is a non-homogeneous system of Linear equations, then the solution is obtained as follows:

**Step 1:** Construct an Augmented matrix $[A/B]$. 
Step 2: Convert \([A/B]\) into row reduced echelon form

Step 3: If \(\rho[A/B] = \rho[A]\), then the system is consistent. Otherwise inconsistent.

Step 4: If \(AX = B\) is consistent, then solution is obtained from the echelon form of \(\rho[A/B]\).

Note: If \(\rho[A/B] = \rho[A] = r\), then there will be \((n - r)\) variables which are Linearly Independent and remaining \(r\) variables are dependent on \((n - r)\) variables

Homogeneous system of Equations

The system of equations \(AX = B\) is said to be homogeneous system of equations if \(B = 0\) i.e. \(AX = 0\).

To obtain solution of homogeneous system of equations the procedure is as follows:

Step 1: Convert \([A]\) into row reduced echelon form

Step 2: Depending on nature of \([A]\), we will solve further.

- \(AX = 0\) is always consistent.
- \(AX = 0\) has a Trivial solution always (i.e. Zero solution)
- If \(\rho[A] = r = n\) (number of variables), then \(AX = 0\) has Unique solution (Trivial solution)
- If \(m = n & |A| \neq 0\) then \(AX = 0\) has only Trivial solution i.e. Zero Solution
- If \(\rho[A] = r < n\) (number of variables (or) unknowns), then \(AX = 0\) has infinitely many solutions.
- If \(m < n\), then \(AX = 0\) has Infinitely many solutions.

Matrix Inversion Method

Suppose \(AX = B\) is a non-homogeneous System of equations, such that \(m = n\) and \(|A| \neq 0\), then

\(AX = B\) has unique solution and is given by \(X = A^{-1}B\)

Cramer’s Rule

Suppose \(AX = B\) is a non-homogeneous System of equations, such that \(m = n\) and \(|A| \neq 0\), then

the solution of \(AX = B\) is obtained as follows:

Step 1: Find determinant of \(A\) i.e. \(|A| = \Delta\) (say)

Step 2: Now, \(x_1 = \frac{\Delta_1}{\Delta}\), where \(\Delta_1\) is the determinant of \(A\) by replacing 1\(^{st}\) column of \(A\) with \(B\).

Step 3: Now, \(x_2 = \frac{\Delta_2}{\Delta}\), where \(\Delta_1\) is the determinant of \(A\) by replacing 2\(^{nd}\) column of \(A\) with \(B\).
**Step 4:** Now, \( x_3 = \frac{\Delta_3}{\Delta} \) where \( \Delta_3 \) is the determinant of \( A \) by replacing 3rd column of \( A \) with \( B \).

\[
\vdots 
\]

Finally \( x_i = \frac{\Delta_i}{\Delta} \) where \( \Delta_i \) is the determinant of \( A \) by replacing \( i \)th column of \( A \) with \( B \).

---

**Gauss Elimination Method**

Let us consider a system of 3 linear equations

\[
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3
\]

The augmented matrix of the corresponding matrix \( A \) is given by \([A|B]\)

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & b_1 \\
a_{21} & a_{22} & a_{23} & b_2 \\
a_{31} & a_{32} & a_{33} & b_3
\end{bmatrix}
\]

Now, our aim is to convert augmented matrix to upper triangular matrix. (i.e. Elements below diagonal are zero).

In order to eliminate \( a_{21} \), multiply with \( \frac{-a_{21}}{a_{11}} \) to \( R_1 \) and add it to \( R_2 \)

\[
\text{i.e.} \quad \left( -\frac{a_{21}}{a_{11}} \right) R_1 + R_2 \Rightarrow \begin{bmatrix}
a_{11} & a_{12} & a_{13} & b_1 \\
a_{21} & a'_{22} & a'_{23} & b'_2 \\
a_{31} & a'_{32} & a'_{33} & b'_3
\end{bmatrix}
\]

Again, In order to eliminate \( a_{31} \), multiply with \( \frac{-a_{31}}{a_{11}} \) to \( R_1 \) and add it to \( R_3 \)

\[
\text{i.e.} \quad \left( -\frac{a_{31}}{a_{11}} \right) R_1 + R_3 \Rightarrow \begin{bmatrix}
a_{11} & a_{12} & a_{13} & b_1 \\
a_{21} & a'_{22} & a'_{23} & b'_2 \\
a_{31} & a''_{32} & a''_{33} & b''_3
\end{bmatrix}
\]

This total elimination process is called as 1st stage of Gauss elimination method.

In the 2nd stage, we have to eliminate \( a''_{32} \). For this multiply with \( \frac{-a''_{31}}{a''_{22}} \) to \( R_2 \) and add it to \( R_3 \)

\[
\text{i.e.} \quad \left( -\frac{a''_{32}}{a''_{22}} \right) R_2 + R_3 \Rightarrow \begin{bmatrix}
a_{11} & a_{12} & a_{13} & b_1 \\
a_{21} & a'_{22} & a'_{23} & b'_2 \\
a_{31} & 0 & a'''_{33} & b'''_3
\end{bmatrix}
\]

Now, above matrix can be written as

\[
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\
a'_{22}x_2 + a'_{23}x_3 = b'_2
\]
From these 3 equations, we can find the value of $x_3, x_2$ and $x_1$ using backward substitution process.

**Gauss Jordan Method**

Let us consider a system of 3 linear equations

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
  a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{align*}
\]

The augmented matrix of the corresponding matrix $A$ is given by $[A|B]$

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & b_1 \\
  a_{21} & a_{22} & a_{23} & b_2 \\
  a_{31} & a_{32} & a_{33} & b_3
\end{bmatrix}
\]

i.e. $\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & b_1 \\
  a_{21} & a_{22} & a_{23} & b_2 \\
  a_{31} & a_{32} & a_{33} & b_3
\end{bmatrix}$

Now, our aim is to convert augmented matrix to upper triangular matrix.

In order to eliminate $a_{21}$, multiply with $-\frac{a_{21}}{a_{21}}$ to $R_1$ and add it to $R_2$

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & b_1 \\
  0 & a'_{22} & a'_{23} & b'_2 \\
  a'_{31} & a'_{32} & a'_{33} & b'_3
\end{bmatrix}
\]

Again, In order to eliminate $a_{31}$, multiply with $-\frac{a_{31}}{a_{31}}$ to $R_1$ and add it to $R_3$

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & b_1 \\
  0 & a'_{22} & a'_{23} & b'_2 \\
  0 & a''_{32} & a''_{33} & b''_3
\end{bmatrix}
\]

This total elimination process is called as 1st stage of Gauss elimination method.

In the 2nd stage, we have to eliminate $a'_{32}$. For this, multiply with $-\frac{a'_{31}}{a'_{22}}$ to $R_2$ and add it to $R_3$

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & b_1 \\
  0 & a'_{22} & a'_{23} & b'_2 \\
  0 & 0 & a''_{33} & b''_3
\end{bmatrix}
\]

In the 3rd stage, we have to eliminate $a_{12}$. For this, multiply with $-\frac{a_{12}}{a'_{22}}$ to $R_2$ and add it to $R_1$

\[
\begin{bmatrix}
  a_{11} & 0 & a'''_{13} & b''''_1 \\
  0 & a'_{22} & a'_{23} & b'_2 \\
  0 & 0 & a''''_{33} & b''''_3
\end{bmatrix}
\]

Now, above matrix can be written as

\[
\begin{align*}
  a_{11}x_1 + a'''_{13}x_3 &= b''''_1 \\
  a'_{22}x_2 + a'_{23}x_3 &= b'_2 \\
  a''''_{33}x_3 &= b''''_3
\end{align*}
\]
From these 3 equations, we can find the value of $x_3, x_2$ and $x_1$ using backward substitution process.

**LU Decomposition (or) Factorization Method (or) Triangularization Method**

This method is applicable only when the matrix $A$ is positive definite (i.e. Eigen values are +ve)

Let us consider a system of 3 linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

The above system of equations can be represented in matrix as follows:

$$
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
=
\begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3
\end{bmatrix}
$$

This is in the form of $AX = B$, where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

If $A$ is positive definite matrix, then we can write $A = LU$, where

$L = \text{Lower triangular matrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \Rightarrow LY = B$

$U = \text{Upper triangular matrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Here, Positive definite $\Rightarrow$ Principle minors are non-zeros

Again, here Principle minors $\Rightarrow$ Left most minors are called as Principle minors

i.e. $[a_{11}], [a_{21} a_{22}]$ etc.

Now, $AX = B \Rightarrow LUX = B \rightarrow 1$

Let $UX = Y \rightarrow 2$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$1 \Rightarrow LY = B \rightarrow 3$

Using Forward substitutions, we get $Y$ from equation $3$
Now, from $2$, R.H.S term $Y$ is known.

Using Backward Substitution get $X$ from $2$ which gives the required solution.

**Solution of Tridiagonal System (Thomas Algorithm)**

Let us consider a system of equations of the form $AX = B$, where

$$ A = \begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 \\ a_1 & b_2 & c_2 & \cdots & 0 \\ 0 & a_2 & b_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & b_{n-1} & c_{n-1} \\ 0 & 0 & \cdots & 0 & a_{n-1} \\ & & & & b_n \end{bmatrix} $$

**Step 1:** Take $\alpha_1 = b_1$

Calculate $\alpha_i = b_i - \frac{a_i c_{i-1}}{\alpha_{i-1}}$, $i = 2, 3, 4, ...$

**Step 2:** Take $\beta_1 = \frac{d_1}{b_1}$

Calculate $\beta_i = \frac{d_i - a_i \beta_{i-1}}{\alpha_i}$, $i = 2, 3, 4, ...$

**Step 3:** Take $x_n = \beta_n$ and

$$ x_i = \beta_i - \frac{c_i x_{i+1}}{\alpha_i}, i = n - 1, n - 2, ..., 1 $$

For Confirmation:

Let $A = \begin{bmatrix} b_1 & c_1 & 0 & \cdots & 0 & 0 \\ a_1 & b_2 & c_2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & c_n \end{bmatrix}$

Now, if we want to make $a_1$ as zero, then $R_2 \rightarrow R_2 - \frac{a_2}{b_1} R_1$. Similarly, we get all other values.

***