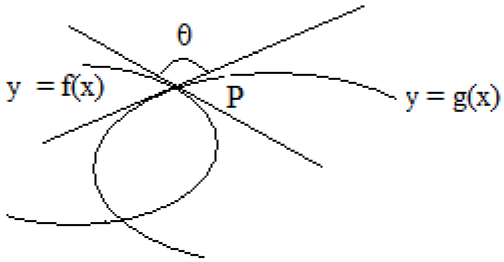
ANGLE BETWEEN TWO CURVES

If two curves intersect at P then the angle between the tangents to the curves at P is called the angle between the curves at P.



Angle between the curves:

Let $y = f(x)$ and $y = g(x)$ be two differentiable curves intersecting at a point P. Let $m_1 = [f'(x)]_P$, $m_2 = [g'(x)]_P$ be the slopes of the tangents to the curves at P. If θ is the acute angle between the curves at P then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

Note 1: If $m_1 = m_2$ then $\theta = 0$. In this case the two curves touch each other at P. Hence the curves have a common tangent and a common normal at P.

Note 2: If $m_1 m_2 = -1$ then $\theta = \frac{\pi}{2}$. In this case the curves cut each other orthogonally at P.

Note 3: If $m_1 = 0$ and $\frac{1}{m_2} = 0$ then the tangents to the curves are parallel to the coordinate axes.

Therefore the angle between the curves is $\theta = \frac{\pi}{2}$.

Exercise

I. Find the angle between the curves given below.

1. $x+y+2=0;\ x^2+y^2-10y=0$

Sol: $x+y+2=0 \Rightarrow x=-(y+2)$ ---- (1)

Equation of the curve $x^2+y^2-10y=0$ -- (2)

Solving above equations, $(y+2)^2+y^2-10y=0$

$\Rightarrow y^2+4y+4+y^2-10y=0$

$\Rightarrow 2y^2-6y+4=0$

$\Rightarrow y^2-3y+2=0 \Rightarrow \quad (y+1)(y-2)=0$

$\Rightarrow y=1$ or $y=2$

$x=-(y+2)$

$y=1 \Rightarrow x=-(1+2)=-3$

$y=2 \Rightarrow x=-(2+2)=-4$

The points of intersection are $P(-3,1)$ and $Q(-4,2)$,

Equation of the curve is $x^2+y^2-10y=0$

Differentiate $x^2+y^2-10y=0$ w.r.to x.

$\Rightarrow 2x+2y\frac{dy}{dx}-10\frac{dy}{dx}=0 \Rightarrow 2\frac{dy}{dx}(y-5)=-2x \Rightarrow \frac{dy}{dx}=-\frac{x}{y-5}$

Equation of the line is $x+y+2=0$

Slope is m_2 = -1.

Case (i):

$\Rightarrow slope\ m_1=\frac{dy}{dx} at P=-\frac{-3}{1-5}=-\frac{3}{4}$ and Slope is m_2 = -1.

Let θ be the angle between the curves, then $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$= \left|\frac{-\frac{3}{4}+1}{1+\frac{3}{4}}\right| = \frac{1}{7} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{7}\right)$$

Case (ii):

$$\Rightarrow slope\ m_1 = \frac{dy}{dx} at Q = -\frac{4}{2-5} = -\frac{4}{3}$$ and Slope is m_2 = -1.

$$\Rightarrow \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-\frac{4}{3}+1}{1+\frac{4}{3}}\right| = \frac{1}{7}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{1}{7}\right)$$

2. $\mathbf{y^2 = 4x \text{ and } x^2 + y^2 = 5}$.

Ans: Points of intersection of $P(1,2)$ and $Q(1,-2)$ and $\theta = \tan^{-1}(3)$

3. $\mathbf{x^2 + 3y = 3 \text{ and } x^2 - y^2 + 25 = 0}$.

Ans: $\theta = \tan^{-1}\left(\dfrac{22\sqrt{6}}{69}\right)$

4. $\mathbf{x^2 = 2(y+1), y = \dfrac{8}{x^2+4}}$

Sol: $x^2 = 2\left(\dfrac{8}{x^2+4}+1\right) = \dfrac{16+2x^2+8}{x^2+4}$

$$\Rightarrow x^2(x^2+4) = 2x^2 + 24$$

$$\Rightarrow x^4 + 4x^2 - 2x^2 - 24 = 0$$

$$\Rightarrow x^2 + 2x^2 - 24 = 0$$

$\Rightarrow (x^2+6)(x^2-4)=0 \Rightarrow x^2=-6 \text{ or } x^2=4$

$x^2=-6 \Rightarrow$ x is not real

$y=\frac{8}{x^2+4}=\frac{8}{4+4}=\frac{8}{8}=1$

$\therefore$ Points of intersection are $P(2,1)$ and $Q(-2,1)$

Equation of the first curve is $x^2=2(y+1)$

$2x=2.\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=x$

$\Rightarrow slope\ m_1=\frac{dy}{dx} at P(2,1)=2$

Equation of the second curve is $y=\frac{8}{x^2+4}$

$\frac{dy}{dx}=\frac{8(-1)}{(x^2+4)^2}-2x=-\frac{16x}{(x^2+4)^2}$

$\Rightarrow slope\ m_2=\frac{dy}{dx} at P(2,1)=\frac{16.2}{(4+4)^2}=\frac{32}{64}=\frac{1}{2}$

$m_1m_2 = 2.-1/2 = -1$

$\therefore$ The given curves cut orthogonally.

Therefore angle between them is $\theta=\frac{\pi}{2}$

Similarly, at $Q(-2,1)$ the angle between the curves is $\theta=\frac{\pi}{2}$

5. $2y^2-9x=0,\ 3x^2+4y=0$ (In the 4th quadrant)

Sol: Given curves are $2y^2-9x=0 \Rightarrow 9x=2y^2 \Rightarrow \ x=\frac{2}{9}y^2$-

Second curve is $3x^2+4y=0$

Solving above equations,

$$\Rightarrow 3.\frac{4}{81}y^4 + 4y = 0$$

$$\Rightarrow \frac{4y^4 + 108y}{27} = 0$$

$$\Rightarrow 4y\left(y^3 + 27\right) = 0$$

$y = 0$ or $y^3 = -27 \Rightarrow y = -3$

$9x = 2y^2 = 2\times 9 \Rightarrow x = 2$

Point of intersection in 4^{th} quadrant is $P(2,-3)$

Equation of the first curve is $2y^2 = 9x$

Differentiate w.r.t. x,

$$4y\frac{dy}{dx} = 9 \Rightarrow \frac{dy}{dx} = \frac{9}{4y}$$

$$\Rightarrow slope\ m_1 = \frac{dy}{dx} at P(2,-3) = \frac{9}{4.-3} = -\frac{3}{4}$$

Equation of the second curve is $3x^2 + 4y = 0$

$\Rightarrow 4y = -3x^2$ differentiate w.r.t x,

$$\Rightarrow 4.\frac{dy}{dx} = -6x$$

$$\Rightarrow \frac{dy}{dx} = \frac{-6x}{4} = \frac{-3x}{2}$$

$$\Rightarrow slope\ m_2 = \frac{dy}{dx} at P(2,-3) = \frac{-3.2}{2} = -3$$

If θ is the angle between the curves then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\theta = \tan^{-1}\left(\frac{9}{13}\right).$

6. $\mathbf{y^2 = 8x, 4x^2 + y^2 = 32}$

Ans: $\theta = \tan^{-1}(3)$ **?**

7. $\mathbf{x^2y = 4,\ y\left(x^2+4\right) = 8}$ **.**

Points $P(2,1), Q(-2,1)$ angle $\theta = \tan^{-1}\left(\frac{1}{3}\right)$

8. Show that the curves $6x^2 - 5x + 2y = 0$ and $4x^2 + 8y^2 = 3$ touch each other at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Sol: Equation of the first curve is $6x^2 - 5x + 2y = 0$

$$\Rightarrow 2y = 5x - 6x^2 \Rightarrow \quad 2.\frac{dy}{dx} = 5 - 12x \Rightarrow \frac{dy}{dx} = \frac{5-12x}{2}$$

$$m_1 = \left(\frac{dy}{dx}\right)_{atP\left(\frac{1}{2},\frac{1}{2}\right)} = \frac{5 - 12.\frac{1}{2}}{2} = \frac{5-6}{2} = -\frac{1}{2}$$

Equation of the second curve is $4x^2 + 8y^2 = 3$

$$\Rightarrow 8x + 16y.\frac{dy}{dx} = 0 \quad \Rightarrow 16y.\frac{dy}{dx} = -8x \quad \Rightarrow \frac{dy}{dx} = \frac{-8x}{16y} = -\frac{x}{2y}$$

$$m_2 = \left(\frac{dy}{dx}\right)_{atP\left(\frac{1}{2},\frac{1}{2}\right)} = \frac{-\frac{1}{2}}{2\left(\frac{1}{2}\right)} = -\frac{1}{2}$$

$\therefore m_1 = m_2$

The given curves touch each other at $P\left(\frac{1}{2}, \frac{1}{2}\right)$.

PROBLEMS FOR PRACTICE

1. Find the slope of the tangent to the following curves at the points as indicated.

i. $y = 5x^2$ at $(-1,5)$ **ii.** $y = 1 - x^2$ at $(2,-3)$

iii. $y = \frac{1}{x-1}$ at $\left(3,\frac{1}{2}\right)$ **iv.** $y = \frac{x-1}{x+1}$ at $(0,-1)$

v. $x = a\sec\theta,\ y = a\tan\theta$ at $\theta = \frac{\pi}{6}$ **vi.** $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ at (a,b).

2. Find the equations of the tangent the normal to the curve $y = 5x^4$ at the point $(1,5)$.

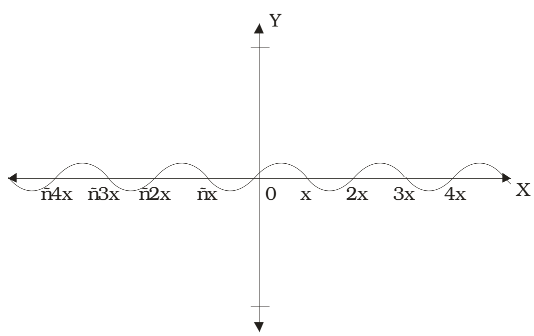
3. Find the equation of the tangent and the normal to the curve $y^4 = ax^3$ at (a,a)

4. Find the equations of the tangent to the curve $y = 3x^2 - x^3$, where it meets the X-axis?

5. Find the points at which the curve $y = \sin x$ horizontal tangents. ?

Sol: $y = \sin x$

$$\frac{dy}{dx} = \cos x$$



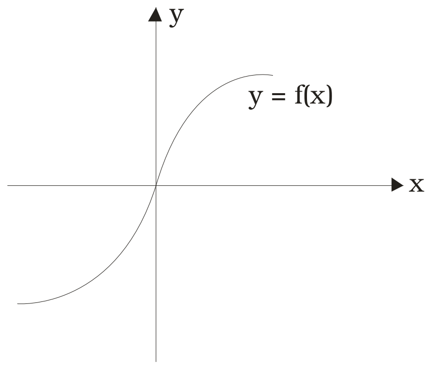
A tangent is horizontal if and arial its slope is $\cos x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}, n \in 2$

Hence the given curve has horizontal tangents at points (x_0, y_0)

$\Leftrightarrow x_0 = (2n+1)\dfrac{\pi}{2}$ and $y_0 = (-1)^n$ for same $n \in z$.

6. Verify whether the curve $y = f(x) = x^{1/3}$ has a vertical tangent at the point with $x = 0$.

Sol:



$$\underset{h\to 0}{Lt}\frac{f(0+h)-f(0)}{h} = \underset{h\to 0}{Lt}\frac{h^{1/3}}{h} - \underset{h\to 0}{Lt}\frac{1}{h^{2/3}}$$

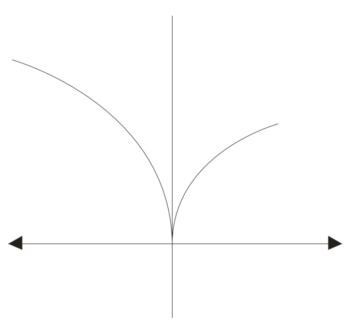
$$\underset{h\to 0}{Lt}\frac{1}{\left(h^{1/3}\right)^2} = \alpha$$

The function has a verified tangent at the point whose x co-ordinate is 0.

7 Find whether the curve $y = f(x) = x^{2/3}$ has a vertical tangent at the point with

x = 0.

Sol:



For $h \neq 0$, we have $\dfrac{f(0+h)-f(0)}{h} = \dfrac{h^{2/3}}{h}$

Thus left handed be normal $\frac{1}{h^{1/3}}$ as $h \to 0$ is $-\alpha$

While the right handed limit is α,

Hence $\underset{h\to 0}{Lt} \frac{1}{h^{1/3}}$ does not exist. The vertical tangent does not exit.

At the point x = 0.

8. Show that the tangent of any point θ on the curve $x = c\sec\theta,\ y = c\tan\theta$ is $y\sin\theta = x - c\cos\theta$.

9. Show that the area of the triangle formed by the tangent at any point on the curve $xy = c\,(c \neq 0)$ with the coordinate axis is constant. ?

Sol: Observe that $c \neq 0$

If c = 0 the equation xy = 0 represents the co-ordinate circle which is against the definite.

Let $P(x_1, y_1)$ be a point on the curve xy = c

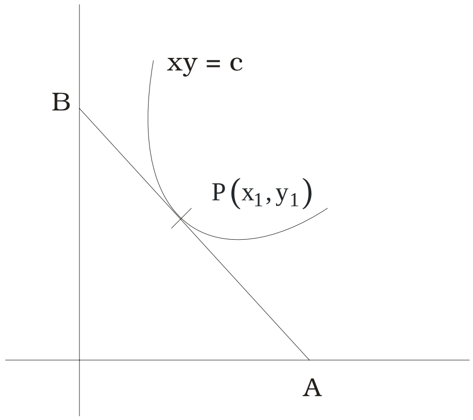
$$y = \frac{c}{x} = 1, \frac{dy}{dx} = -\frac{c}{x^2}$$

Equation of the tangent at (x_1, y_1) is

$$y - y_1 = -\frac{c}{x^2}(x - x_1)$$

$$\Rightarrow x^2 y - x_1^2 y_1 = -\alpha + \alpha_1$$

$$\Rightarrow \alpha + x_1^2 . y = x_1^2 + \alpha_1 \quad = \alpha_1 + \alpha_1 (x_1 y_1 = c) \quad = 2\alpha_1$$



$$\frac{\alpha}{2\alpha_1}+\frac{x_1^2.y}{2\alpha_1}=1 \Rightarrow \qquad \frac{x}{2x_1}+\frac{y}{\left(\frac{2c}{x_1}\right)}=1$$

Area of the triangle formed with co-ordinate axes $=\frac{1}{2}|OA.OB|$

$$=\frac{1}{2}(2x_1)\left(\frac{2c}{x_1}\right)=2c=\text{constant}.$$

10. Show that the equation of the tangent to the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2(a\neq 0, b\neq 0)$ at the point (a,b) is $\frac{x}{a}+\frac{y}{b}=2$.

Sol: Equation of the curve is $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$

Differentiating w.r.to x_1 we get

$$n\left(\frac{x}{a}\right)^{n-1}.\frac{1}{a}+n\left(\frac{y}{b}\right)^{n-1}.\frac{1}{b}.\frac{dy}{dx}=0$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(a,b)}=\frac{\left(-\frac{n}{a}\right)\left(\frac{a}{a}\right)^{x-1}}{\left(\frac{n}{b}\right)\left(\frac{b}{b}\right)^{x-1}}=-\frac{b}{a}$$

Equation of the tangent to the curve at the point (a,b) is

$$y-b=-\frac{b}{a}(x-a)\Rightarrow \qquad \frac{y}{b}-1=-\frac{x}{a}+1\Rightarrow \qquad \frac{x}{a}+\frac{y}{b}=2$$

11. If the line $x\cos\alpha + y\sin x\alpha = p$ touches the curve $\left(\frac{x}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y}{b}\right)^{\frac{n}{n-1}=1}$ then show that $p^n = (a\cos\alpha)^n + (b\sin\alpha)^n$.

12. If the normal at the curve $ay^2 = x^3\ (a \neq 0)$ at a point makes equal intercepts with the co-ordinate axes, then find the x co-ordinate of the point

Sol: Let $P(x_1, y_1)$ be the point on the curve $ay^2 = x^2$

Differentiating w.r.to x

$$2ay = \frac{dy}{dx} = 3x^2 \Rightarrow \left(\frac{dy}{dx}\right) = \frac{3x_1^2}{2ay_1}$$

Equation of the normal to the curve at a $y - y_1 = -\frac{2ay_1}{3{x_1}^2}(x - x_1)$

$$3{x_1}^2 y - 3{x_1}^2 y_1 = -2ay_1 x + 2ax_1 y_1$$

$$\Rightarrow 2ay_1 x + 3{x_1}^2 y = 2ax_1 y_1 + 3{x_1}^2 y_1$$

$$\frac{x}{\left(\frac{3{x_1}^2 y_1 + 2ax_1 y_1}{2ay_1}\right)} + \frac{y}{\left(\frac{3x_1^2 y_1 + 2ax_1 y_1}{3{x_1}^2}\right)} = 1$$

Given $\frac{3{x_1}^2 + 2ay_1}{2ay_1} = \frac{3x_1^2 y_1 + 2ax_1 y_1}{3{x_1}^2}$

$$3{x_1}^4 = 2ay_1 \Rightarrow 9{x_1}^4 = 4a^2 y_1^2$$

But $ay_1^2 = x_1^3$

$$\therefore 9x_1^4 = 4ax_1^3 \Rightarrow \quad x_1^3(9x_1 - 4a) = 0 \Rightarrow \quad x_1 = 0\ (\text{or})\ \frac{4a}{9}$$

13. The tangent to the curve $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)(a \neq 0)$ at a point P on it is parallel to x-axis. Prove that all such points P lie on the curve $y^2 = 4ax$

Sol: Equation the curve $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$

Differentiating w.r.to x

$$2y.\frac{dy}{dx} = 4a\left(1 + \cos\frac{x}{a}\right) \Rightarrow \frac{dy}{dx} = \frac{2a}{y}\left(1 + \cos\frac{x}{a}\right)$$

$P(x_1, y_1)$ be a point o the curve at which the tangent parallel to x-axis.

$\therefore$ Slope of the tangent is zero

$$\Rightarrow \left(\frac{dx}{dy}\right)P = 0 \Rightarrow \frac{2a}{y}\left(1 + \cos\frac{x_1}{a}\right) = 0$$

$$1 + \cos\frac{x_1}{a} = 0 \Rightarrow \cos\frac{x_1}{a} = -1$$

$$\sin\frac{x_1}{a} = 0 \qquad \text{--- (1)}$$

$P(x_1, y_1)$ lies on the given curve $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$

$$\Rightarrow y_1^2 = 4a\left(x_1 + a\sin\frac{x_1}{a}\right) = 4ax_1 + 0 \text{ by } (1)$$

i.e., $y_1^2 = 4ax_1$

$\therefore$ P lies on the curve $y^2 = 4ax$

14. Show that the length of the sub-normal at any point of the curve $y^2 = 4ax$ is a constant.

15. Show that the length of the sub tangent at any point on the curve $y = a^x\ (a > 0)$ is constant a.

16. Show that the square of the length of sub tangent at any point on the curve $by^2 = (x+a)^2 \, (b \neq 0)$ varies as the square of the length of the sub-normal at that point.

Sol: Length of the curve is $by^2 = (x+a)^2$

Differentiating w.r.to x

$$2by\frac{dy}{dx} = 3(x+a)^2$$

L.N= length of the subnormal of any point $p(x,y) = \left|y.\frac{dy}{dx}\right|$

$$= \left|y.\frac{3(x+a)^2}{2by}\right| = \frac{3(x+a)^2}{2b}$$

L.T=length of the sub tangent

$$= \left|\frac{y}{\left(\frac{dy}{dx}\right)}\right| = \left|y.\frac{2by}{3(x+a)^2}\right| = \frac{2.by^2}{3(x+a)^2} = \frac{2(x+a)^3}{3(x+a)^2} = \frac{2}{3}(x+a)$$

$$\frac{L.N}{L.T^2} = \frac{3(x+a)^2}{2b}.\frac{9}{4(x+a)^2} = \frac{27}{8b}$$

Square of the length of sub tangent at any point on the curve varies as the square of the length of the sub-normal .at

17. Find the value of k, so that the length of the subnormal at any point on the curve $y = a^{1-k}.x^k$ is constant.

18. Show that at any point o the curve $x^{m+n} = a^{m-n}.y^{2n} \, (a > 0, m+n \neq 0)$ m^{th} power of the length of the sub tangent varies of the n^{th} power of length of the sub-normal.

19. Find the angle between the curve $xy = z$ and $x^2 + 4y = 0$.

20. Find the angle between the curve $2y = e^{\frac{-x}{2}}$ and y-axis.

Sol: Equation of y-axis is x = 0

The point of intersection of the curve $2y = e^{\frac{-x}{2}}$ and $x = 0$ is $P\left(0, \frac{1}{2}\right)$

The angle ψ made by the tangent to the curve $2y = e^{\frac{-x}{2}}$ at P with $x-$axis is given by

$$\tan\psi = \left.\frac{dy}{dx}\right|_{\left(0,\frac{1}{2}\right)} = \left.\frac{-1}{4}e^{\frac{-x}{2}}\right|_{\left(0,\frac{1}{2}\right)} = \frac{-1}{4}$$

Further, if ϕ is the angle between the $y-$axis and $2y = e^{\frac{-x}{2}}$, then we have

$$\tan\phi = \left|\tan\left(\frac{\pi}{2}-\psi\right)\right| - |\cot\psi| = 4$$

$\therefore$ The angle between the curve and the y-axis is $\tan^{-1}4$.

21. Show that the condition of the orthogonally of the curves $ax^2 + by^2 = 1$ and $a_1x^2 + b_1y^2 = 1$ is $\frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}$.

Sol: Let the curves $ax^2 + by^2 = 1$ and $a_1x^2 + b_1y^2 = 1$ intersect at $p(x_1, y_1)$ so that $ax_1^2 + by_1^2 = 1$ and $a_1x_1^2 + b_1y_1^2 = 1$, from which we get,

$$\frac{x_1^2}{b_1 - b} = \frac{y_1^2}{a_1 - a} = \frac{1}{ab_1 - a_1b} \qquad \text{--- (1)}$$

Differentiating $ax^2 + by^2 = 1$ with respect to x, we bet $\frac{dy}{dx} = \frac{-ax}{by}$

Hence, if m_1 is the slope of the tangent at $P(x_1, y_1)$ to the curve

$$ax^2 + by^2 = 1, m_1 = \frac{-ax_1}{by_1}$$

Similarly, the slope (m_2) of the tangent at P to $a_1x^2 + b_1y^2 = 1$ is given by $m_2 = \frac{-a_1x_1}{b_1y_1}$

Since the curves cut orthogonally we have $m_1m_2 = -1$.

i.e., $\frac{aa_1x_1^2}{bb_1y_1^2} = -1$ or $\frac{x_1^2}{y_1^2} = -\frac{-bb_1}{aa_2}$ --- (2)

Now from (1) and (2), the condition for the orthogonally of the given curves is

$$\frac{b_1 - b}{a - a_1} = \frac{bb_1}{aa_1}$$

Or $(b - a)a_1b_1 = (b_1 - a_1)ab$

Or $\frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}$