4. PERMUTATIONS AND COMBINATIONS

Quick Review

1. An arrangement that can be formed by taking some or all of a finite set of things (or objects) is called a permutation.

2. A permutation is said to be a linear permutation if the objects are arranged in a line. A linear permutation is simply called as a permutation.

3. A permutation is said to be a circular permutation if the objects are arranged in the form of a circle (a closed curve).

4. The number of (linear) permutations that can be formed by taking r things at a time from a set of n dissimilar things (r ≤ n) is denoted by \( ^nP_r \) or \( P(n, r) \) or \( P_{\binom{n}{r}} \).

5. The number of permutations of n dissimilar things taken r at a time is equal to the number of ways of filling of r blank places arranged in a row by n dissimilar things.

6. Fundamental counting principle: If an operation can be performed in m ways and a second operation can be performed in n ways corresponding to each performance of the first operation, then the two operations in succession can be performed in mn ways.

7. If in k operations, the first operation can be performed in \( n_1 \) ways, the second operation can be performed in \( n_2 \) ways, third operation can be performed in \( n_3 \) ways and so on, then the k operations in succession can be performed in \( n_1 n_2 n_3 \ldots n_k \) ways.

8. \( ^nP_r = n(n – 1)(n – 2)\ldots(n – r + 1) \).

9. If \( n \) is a non-negative integer, then factorial \( n \) is denoted by \( n! \) or \( \angle n \) and defined as follows.
   (i) \( 0! = 1 \);
   (ii) \( \text{If } n > 0 \text{ then } n! = n \times (n – 1)! \).

10. If \( n \) is a positive integer, then \( n! \) is the product of first \( n \) positive integers. i.e., \( n! = 1.2.3\ldots n \).

11. \( ^nP_r = \frac{r!}{(n-r)!} \).

12. The number of permutations of n dissimilar things taken all at a time is \( ^nP_n = n! \).

13. \( ^nP_r = (n-1)^{P_r} + r(n-1)^{P_{r-1}} \).

14. The number of injections (one-one functions) that can be defined from a set containing \( r \) elements into a set containing \( n \) elements is \( ^nP_r \).

15. The number of bijections (one-one onto functions) that can be defined from a set containing \( n \) elements onto a set containing \( n \) elements is \( n! \).

16. The number of permutations of \( n \) dissimilar things taken \( r \) at a time when repetition of things is allowed any number of times is \( n^r \).

17. The number of permutations of \( n \) dissimilar things taken not more than \( r \) at a time, when each thing may occur any number of times is \( \frac{n(n^r - 1)}{n - 1} \).

18. The number of functions that can defined from a set containing \( r \) elements into a set containing \( n \) elements is \( n^r \).
19. The number of permutations of \( n \) things taken all at a time when \( p \) of them are all alike and the rest all different is \( \frac{n!}{p!} \).

20. If \( p_1 \) things are alike of one kind, \( p_2 \) things are alike of second kind, \( p_3 \) things are alike of third kind and so on, \( p_k \) things are alike of \( k^{th} \) kind in \( p_1+p_2+\ldots+p_k \) things, then the number of permutations obtained by taking all the things is \( \frac{(p_1+p_2+\ldots+p_k)!}{p_1!p_2!\ldots p_k!} \).

21. The number of circular permutations on \( n \) different things taken \( r \) at a time is \( \frac{n^r}{r} \).

22. The number of circular permutations of \( n \) different things taken all at a time is \( (n-1)! \).

23. The number of circular permutations of \( n \) things taken \( r \) at a time in one direction is \( \frac{n^r}{r} \).

24. The number of circular permutations of \( n \) things taken all at a time in one direction is \( \frac{1}{2}(n-1)! \).

25. (i) \( n^r P_r + r^r P_{r-1} = \frac{n^{n+1}}{r} \)
(ii) \( \frac{a^r}{a-1} P_{r-1} = n 
(iii) \( \frac{n^r}{r} P_{r-1} = n-r+1 \).

26. If \( n \) is a positive integer and \( p \) is a prime number then the exponent of \( p \) in \( n! \) is \( \left[ \frac{n}{p} \right] + \left[ \frac{n}{p^2} \right] + \left[ \frac{n}{p^3} \right] + \ldots \) where \([x]\) denotes the greatest integer \( \leq x \).

27. The number of ways in which \( m \) (first type of different) things and \( n \) (second type of different) things \( (m+1 \geq n) \) can be arranged in a row so that no two things of second kind come together is \( m! \times (n+1)! \).

28. The number of ways in which \( m \) (first type of different) things and \( n \) (second type of different) things can be arranged in a row so that all the second type of things come together is \( n! \times (m+1)! \).

29. The number of ways in which \( n \) (first type of different) things and \( n \) (second type of different) things can be arranged in a row alternatively is \( 2 \times n! \times n! \).

30. Sum of the numbers formed by taking all the given \( n \) digits (excluding 0) is \( (\text{Sum of all the } n \text{ digits}) \times (n-1)! \times (111\ldots n \text{ times}) \).

31. Sum of the numbers formed by taking all the given \( n \) digits (including 0) is \( (\text{Sum of all the } n \text{ digits}) \times [(n-1)! \times (111\ldots n \text{ times}) - (n-2)! \times (111\ldots(n-1) \text{ times})] \).

32. Sum of all \( r \)-digit numbers formed by taking the given \( n \) digits (without zero) is \( (\text{sum of all the } n \text{ digits}) \times n^r P_{r-1} \times (111\ldots r \text{ times}) \).

33. Sum of all the \( r \)-digit numbers formed by taking the given \( n \) digits (including 0) is \( (\text{sum of all the } n \text{ digits}) \times n^r P_{r-1} \times (111\ldots r \text{ times}) - n^r P_{r-2} \times (111\ldots(r-1) \text{ times}) \).
34. The number of ways in which m (first type of different) things and n (second type of different) things, \( m \geq n \) can be arranged in a circle so that no two things of second kind come together is \((m - 1)! \, ^{m}P_{n}\).

35. The number of ways in which m (first type of different) things and n (second type of different) things can be arranged in a circle so that all the second type of things come together is \(m! \, \frac{n!}{n-1}\).

36. The number of ways in which m (first type of different) things and n (second type of different) things can be arranged in the form of garland so that all the second type of things come together is \(m! \, \frac{n!}{2}\).

37. A selection that can be formed by taking some or all of a finite set of things (or objects) is called a combination.

38. Formation of a combination by taking r elements from a finite set A means picking up an r element subset of A.

39. The number of combinations of n dissimilar things taken r at a time is equal to the number of r element subsets of a set containing n elements.

40. The number of combinations of n dissimilar things taken r at a time is denoted by \( ^{n}C_{r} \) or \( C(n, r) \) or \( \binom{n}{r} \) or \( \frac{n!}{r!(n-r)!} \).

41. \( ^{n}C_{r} = \frac{n!}{r!(n-r)!} \).

42. \( ^{n}C_{r} = \frac{n \cdot (n-1) \cdot (n-2) \ldots (n-r+1)}{r! \cdot 1 \cdot 2 \cdot 3 \ldots r} \).

43. \( ^{n}C_{r} = ^{n}C_{n-r} \).

44. \( ^{n}C_{r} + \frac{n}{r} \cdot ^{n}C_{r-1} = \frac{(n+1)}{r} \cdot ^{n+1}C_{r} \).

45. If \( ^{n}C_{r} = ^{n}C_{s} \), then \( r = s \) or \( r + s = n \).

46. The number of diagonals in a regular polygon of n sides is \( ^{n}C_{2} = \frac{n(n-3)}{2} \).

47. The number of ways in which \( m + n \) things can be divided into two different groups of m and n things respectively is \( \frac{(m+n)!}{m!n!} \).

48. The number of ways in which 2n things can be divided into two equal groups of n things each is \( \frac{(2n)!}{2n \cdot (n!)^2} \).

49. The number of ways in which \( n_{1} + n_{2} + \ldots + n_{k} \) things can be divided into k different groups of \( n_{1} \) things, \( n_{2} \) things, \( n_{3} \) things, \ldots \ldots \ldots \) \( n_{k} \) things respectively is \( \frac{(n_{1} + n_{2} + \ldots + n_{k})!}{n_{1}!n_{2}!\ldots n_{k}!} \).

50. The number of ways in which \( kn \) things can be divided into k equal groups of n things each is \( \frac{(kn)!}{k!(n!)^k} \).

51. The total number of combinations of \( p + q \) things taken any number at a time when p things are alike of one kind and q things are alike of a second kind is \( (p + 1) \cdot (q + 1) \).
52. The total number of combinations of \( p + q \) things taken any number at a time, includes the case in which nothing will be selected.

53. The total number of combinations of \( (p + q) \) things taken one or more at a time when \( p \) things are alike of one kind and \( q \) things are alike of a second kind is \( (p + 1)(q + 1) – 1 \).

54. The total number of combinations of \( (p_1+p_2+...+p_k) \) things taken any number at a time when \( p_1 \) things are alike of one kind, \( p_2 \) things are alike of a second kind, ... \( p_k \) things are alike of kth kind, is \( (p_1 + 1)(p_2 + 1) ... (p_k + 1) \).

55. The total number of combinations of \( (p_1+p_2+...+p_k) \) things taken one or more at a time when \( p_1 \) things are alike of one kind, \( p_2 \) things are alike of a second kind, ... \( p_k \) things are alike of kth kind, is \( (p_1 + 1)(p_2 + 1) ... (p_k + 1) – 1 \).

56. The total number of combinations of \( n \) different things taken any number at a time is \( n^2 \).

57. The total number of combinations of \( n \) different things taken one or more at a time is \( n^2 – 1 \).

58. \( ^nC_0 + ^nC_1 + ^nC_2 + ... ^nC_n = 2^n \).

59. If \( n \) is a positive integer, then \( n \) can be uniquely expressed as \( n = p_1^{a_1}p_2^{a_2}...p_k^{a_k} \) where \( p_1, p_2, ..., p_k \) are primes in increasing order and \( a_1, a_2, ..., a_k \) are non-negative integers. This representation of \( n \) is called prime factorisation of \( n \) in canonical form or prime power factorisation of \( n \).

60. The number of positive divisors of a positive integer \( n = p_1^{a_1}p_2^{a_2}...p_k^{a_k} \) (the prime factorisation) is \( (a_1 + 1)(a_2 + 1)(a_3 + 1)...(a_k + 1) \).

61. (i) \( r^nC_r = r^nC_{n-r} \) (ii) \( ^nC_r + ^nC_{r-1} = \binom{n}{r+1} \) (iii) \( ^nC_r = ^nC_s \Rightarrow r = s \) or \( r + s = n \)

62. (i) \( \frac{n!}{(n-r)!r!} \) (ii) \( \frac{n!}{(n-r)!r!} \) (iii) \( \frac{n!}{(n-r)!r!} = \frac{n}{r} \)

63. The number of parallelograms formed when a set of \( m \) parallel lines are intersecting another set of \( n \) parallel lines is \( ^mC_2 \times ^nC_2 \).

64. If there are \( n \) points in a plane no three of which are on the same straight line excepting \( p \) points which are collinear, then

(i) the number of straight lines formed by joining them is \( ^nC_2 - ^nC_2 + 1 \).

(ii) the number of triangles formed by joining them is \( ^nC_3 - ^nC_3 \).

65. The number of ways that \( n \) sovereigns can be given away when there are \( k \) applicants and any applicant may have either 0, 1, 2, 3, 4, 5 ... ... or \( n \) sovereigns is \( ^nC_k \).

66. (i) The number of ways in which exactly \( r \) letters can be placed in wrongly addressed envelopes when \( n \) letters are putting in \( n \) addressed envelopes is \( n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... + (-1)^r\frac{1}{r!}\right] \).

(ii) The number of ways in which \( n \) different letters can be put in their \( n \) addressed envelopes so that all the letters are in the wrong envelopes is \( n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... + (-1)^n\frac{1}{n!}\right] \).

67. (i) The number of ways of answering one or more of \( n \) questions is \( 2^n - 1 \).

(ii) The number of ways of answering one or more of \( n \) questions when each question have an alternative is \( 3^n - 1 \).
(iii) The number of ways of answering all of \( n \) questions when each question have an alternative is \( 2^n \).

68. The number of distinct positive integral divisors of \( p_1^{k_1}p_2^{k_2}...p_r^{k_r} \) where \( p_1, p_2, ..., p_r \) are primes in ascending order, is \( (k_1 + 1)(k_2 + 1) \cdots (k_r + 1) \).

69. The sum of distinct positive integral divisors of \( p_1^{k_1}p_2^{k_2}...p_r^{k_r} \) where \( p_1, p_2, ..., p_r \) are primes in ascending order, is

\[
\frac{p_1^{k_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{k_2+1} - 1}{p_2 - 1} \cdots \frac{p_r^{k_r+1} - 1}{p_r - 1}.
\]