ROTATORY MOTION

Synopsis :

CIRCULAR MOTION :

1. In **translatory motion**, every particle travels the same distance along parallel paths, which may be straight or curved. Every particle of the body has the same velocity and acceleration.
2. In **rotatory motion,** the body rotates about a fixed axis. Every particle of the body describes a circular path and centres of concentric circles lie on the axis of rotation. Every particle of the body undergoes the same angular displacement. But linear velocities of rotating particles differ depending upon their radii of rotation.
3. When a particle describes a circular path, the line joining the centre of the circle and the position of the particle at any instant of time is called the **radius vector**.
4. As the particle moves round the circle, the radius vector rotates (like the hands of a clock). The angle described by the radius vector in a given interval of time is called the **angular displacement**.
5. Angular displacement is a vector passing through the centre and directed along the perpendicular to the plane of the circle whose direction is determined by right hand screw rule (It is a pseudo vector).
6. Angular displacement is measured in radians or turns.
7. The rate of change of angular displacement is called **angular velocity** (ω).

   $\omega = \frac{\theta}{t}$ rads^{-1} or rpm; $1\,\text{rpm} = \frac{2\pi}{60} = \frac{\pi}{30}$ rads^{-1}.
8. Angular velocity is a vector lying in the direction of angular displacement.
9. Linear velocity $(\vec{V}) = \vec{\omega} \times \vec{r}$.
10. Rate of change of angular velocity is called **angular acceleration** (α). Unit is rads$^{\theta 2}$.

    $\alpha = \frac{\text{change in angular velocity}}{\text{time}}$.
11. Linear acceleration = radius × angular acceleration. $\vec{a} = \vec{\alpha} \times \vec{r}$.
12. Resultant acceleration a = ω where a_r = radial acceleration and a_T = tangential acceleration.
13. Angular displacement (θ), angular velocity (ω) and angular acceleration (α) are pseudo vectors.
14. For a body rotating with uniform angular acceleration, the following equations hold good.

    i) $\omega = \omega_0 + \alpha t$.......similar to $v = u + at$

    ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ similar to $s = ut + \frac{1}{2}at^2$

    iii) $\omega^2 - \omega_0^2 = 2\alpha\theta$ similar to $v^2 - u^2 = 2as$

    iv) W = $\tau\theta$ similar to W = Fs

    v) Power = $\tau\omega$ similar to P = Fv

    vi) Torque (τ) = $I\alpha$ similar to F = ma

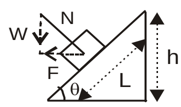
    vii) Angular momentum (L) = $I\omega$ similar to P = mv

    viii) Rotational kinetic energy = $\frac{1}{2}I\omega^2$ similar to $K.E. = \frac{1}{2}mv^2$
15. Angular momentum and torque are pseudo vectors.

16. When a body is moving in a circular path with uniform speed, then the acceleration experienced by the body, along the radius of the circle and directed towards the centre is called **normal** or **radial** or **centripetal acceleration** and is equal to $\frac{v^2}{r}$ or $r\omega^2$ or $4\pi^2 n^2 r$ where n is the number of revolutions per second.
17. The force which makes a body move round a circular path with uniform speed is called the **centripetal force**. This is always directed towards the centre of the circle.
18. If the direction of a force of constant magnitude applied on a body is always at right angles to the direction of its motion, then it describes a circular path with a uniform speed and its kinetic energy remains constant.
19. For a body moving round a circular path with uniform speed, the time period of revolution $T = \frac{2\pi}{\omega}$ and frequency $n = \frac{1}{T} = \frac{\omega}{2\pi}$ i.e., $\omega = 2\pi n$ where n is the number of revolutions per second.
20. Centripetal force = $\frac{mv^2}{r} = mr\omega^2$.
21. A body moving round a circular path with uniform speed experiences an inertial or pseudo force which tends to make it go away from the centre. This force is called the **centrifugal force** and this is due to the inertia of the body.
22. Centrifugal force = –centripetal force (but these are not action and reaction).
23. No work is done by centripetal force.
24. The kinetic energy of the body revolving round in a circular path with uniform speed is 'E'. If 'F' is the required centripetal force, then

$$F = \frac{2E}{r}$$

25. Uses of centrifugal forces and centrifugal machines.
    i) Cream is separated from milk (cream separator)
    ii) Sugar crystals are separated from molasses.
    iii) Precipitate is separated from solution.
    iv) Steam is regulated by Watt's governer.
    v) Water is pumped from a well (Electrical pump).
    vi) Hematocentrifuge, Grinder, Washing machine, etc.
26. The angle through which a cyclist should lean while taking sharp turnings is given by the relation $\theta = \mathrm{Tan}^{-1}\left(\frac{v^2}{rg}\right)$.
27. Safe speed on an unbanked road when a vehicle takes a turn of radius r is $v = \sqrt{\mu rg}$ where μ = coefficient of friction.
28. The maximum speed that is possible on curved unbanked track is given by $g = v^2h/ar$ where h = height of centre of gravity and a = half the distance between wheels.
29. After banking of a road, the weight W of the vehicle, the normal reaction N and the centripetal force F form a vector triangle (or) centripetal force is the resultant of W and N. Angle of banking θ is given by $\tan\theta = v^2/rg$ and the height of banking is given by $h = L\sin\theta$ where L is the width of the road.

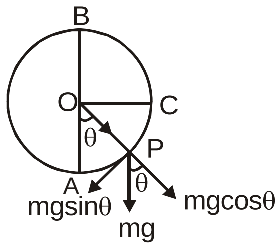


If θ is very small, then $\theta = \frac{v^2}{rg} = \frac{h}{L}$

30. When a mixture of liquids of different densities is centrifuged, the denser liquid moves as far away from the axis as possible.

MOTION IN A VERTICAL CIRCLE :

31. A particle of mass 'm' suspended by a thread is given a horizontal speed 'u'. When it is at 'A',it moves in a vertical circle of radius 'r'



32. **When the displacement of the particle is 'θ', i.e when the particle is at p**
   a) Speed of the particle
      $v = \sqrt{u^2 - 2gr(1 - Co\theta)}$ where u is the velocity at A, the lowest point
   b) Centripetal force $mv^2 / r = T - mg \cos\theta$
   c) The speed of the particle continuously changes. It increase while coming down and decreases while going up
   d) This is an example for non - uniform circular motion.
   e) Tangential acceleration = $g \sin\theta$
   f) Tangential force = $mg \sin\theta$
   g) Tension in the string $T = mv^2/r + mg \cos\theta = \frac{m}{r}[v^2 + g(r - h)]$
   h) Velocity, speed, K.E, linear momentum, angular momentum, angular velocity, all are variables. Only total energy remains constant

33. If $u < \sqrt{2gr}$ the body oscillates about A.
   $0 < u < \sqrt{2gr}$; $0 < \theta < 90°$.
34. If $\sqrt{2gr} < u < \sqrt{5gr}$ the body leaves the path without completing the circle.
35. A body is projected with a velocity 'u' at the lowest point
   a) Height at which velocity u = 0. is $h = u^2/2g$
   b) Height at which Tension T = 0 is
      $h = \frac{u^2 + rg}{3g}$
   c) Angle with vertical at which velocity v = 0. is $\cos\theta = 1 - \frac{u^2}{2gr}$
   d) Angle with vertical at which the tension
      T = 0 is $\cos\theta = 2/3 - u^2/3gr$
   e) Tension in the string at an angular displacement θ with vertical is
      $T = mu^2/r - mg(2 - 3\cos\theta)$

36. When the body is projected horizontally with a velocity $u = \sqrt{5gr}$ from the lowest point A,
   a) It completes the circle
   b) Velocity at the top B = $\sqrt{gr}$ called the critical speed
   c) Tension in the string at the top $T_1 = 0$
   d) Tension in the string at the lowest point
      $T_2 = 6mg$. It is the maximum tension in the string)
   e) $T_2 - T_1 = 6mg$.
   f) Velocity at the horizontal position ie, at C $V_e = \sqrt{3gr}$
   g) Tension in the string at C = 3mg

h) Ratio of velocities at A, B and C = $\sqrt{5} : \sqrt{3} : \sqrt{1}$

i) Ratio of K.E at A, B and C = 5: 3: 1

j) Velocity at an angular displacement θ is given by V = $\sqrt{3gr + 2gr\cos\theta}$

k) V_{min} does not depend on the mass of the body

l) Tension at angular displacement 'θ' is given by T = 3mg (1+Cosθ)

m) T_{min} in the string does not depend on the radius of vertical circle.

37. When the body is rotated at a constant speed, 'v'

a) Tension in the string at the lowest point
$T = mv^2/r + mg$

b) Tension in the string at the highest point
$T = mv^2/r - mg$

c) Tension in the string at the horizontal position
$T = mv^2/r$

d) Time period of revolution if v = $\sqrt{rg}$ is
$T = 2\pi\sqrt{r/g}$

38. A sphere of mass m is suspended from fixed point by means of light string. The sphere is made to move in a vertical circle of radius r whose centre coincides with point of suspension such that the velocities of sphere are minimum or critical at different points. Then

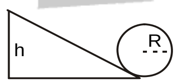
at the lowest point $K.E_{max} = \frac{5}{2}mgr$

at the highest point = $K.E_{min} = \frac{1}{2}mgr$

during one revolution ΔK.E = ΔP.E = 2mgr

39. A ball of mass 'M' is suspended vertically by a string of length 'l'. A bullet of mass 'm' is fired horizontally with a velocity 'u' on to the ball sticks to it. For the system to complete the vertical circle, the minimum value of 'u' is given by $u = \frac{M+m}{m}\sqrt{5gl}$.

40. A body of mass m is sliding along an inclined plane from a vertical height h as shown in the figure. For the body to describe a vertical circle of radius R, the minimum height in terms of R is given by $h = \frac{5R}{2}$.



41. a) A particle is freely sliding down from the top of a smooth convex hemisphere of radius r. The particle is ready to leave the surface at a vertical distance h = r/3 from the highest point.

b) If the position vector of the particle with respect to the centre of curvature makes an angle θ with vertical then Cosθ = 2/3

42. During non–uniform acceleration, $\vec{a}_T = \vec{\alpha} \times \vec{r}$ and $\vec{a}_R = \vec{\omega} \times \vec{v}$ where $\vec{\alpha}$ is angular acceleration, $\vec{a}_T$ is tangential (linear) acceleration and $\vec{a}_R$ is radial or centripetal acceleration.

43. **Moment of inertia (I)** of a body about an axis is defined as the sum of the products of the masses and the squares of their distances of different particles from the axis of rotation.

44. **Moment of inertia or Rotational inertia**

a) It is the property of a body due to which it opposes any change in its state of rest or uniform rotation.

b) It is the rotational analogue of inertia in translatory motion

c) For a particle of mass 'm' rotating at a distance r from the axis of rotation. $I = mr^2$
d) For a rigid body $I = mk^2$ K is called radius of gyration
e) Radius of gyration is the distance whose square when multiplied by mass of the body gives moment of inertia of the body about the given axis.

$$K = \sqrt{I/m}$$

f) S.I unit of moment of inertia is $Kg.m^2$.
g) Moment of inertia of a body depends on
   i) Mass of the body
   ii) Distribution of mass of the body
   iii) Position of axis of rotation
   iv) Temperature of the body
h) It is independent of angular velocity of rotation of the body.

45. The root mean square of the distance of all the particles, from the axis of rotation is known as **radius of gyration (K)**.

$$K = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + ... + r_n^2}{n}}$$

46. The radius of gyration of a rigid body about a given fixed axis is that perpendicular distance from the given axis where the entire mass of the body can be redistributed and concentrated without altering the moment of inertia of the body about the given axis.

$$I = MK^2 \text{ or } K = \sqrt{I/M}$$

47. Two small spheres of masses m_1 and m_2 are joined by a rod of length 'r' and of negligible mass. The moment of inertia of the system about an axis passing through the centre of mass and perpendicular to the rod, treating the spheres as particles is

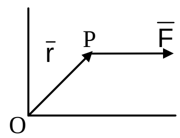
$$I = \left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2$$

48. **Perpendicular axes theorem :** The moment of inertia of a plane lamina about an axis perpendicular to its plane is the sum of the moments of inertia of the same lamina about two mutually perpendicular axes, lying in the plane of the lamina and intersecting on the given axis.
$I_z = I_x + I_y$

49. **Parallel axes theorem :** The moment of inertia of any rigid body about any axis is equal to the moment of inertia of the same body about a parallel axis passing through its centre of mass plus the product of the mass of the body and square of the distance between the parallel axes. $I = I_G + Md^2$.

50. **Angular momentum ($\vec{L}$) :**
(i) The moment of linear momentum is called angular momentum of the particle about the axis of rotation.
(ii) $\vec{L} = mvr = mr^2\omega = I\omega$
(iii) It is a vector quantity.
(iv) SI unit is kgm^2s^{-1} or Js
(v) $\vec{L} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$

51. **Torgue**



a) A force $\bar{F}$ acting on a particle at p whose position vector is $\bar{r}$. Then the torque $\bar{F}$ about 'O' is defined as $\bar{\tau} = \bar{r} x \bar{F}$
b) It is an axial vector. Its direction is given by right hand thumb rule.
c) S.I unit is N.m

d) $\tau = Fr \sin\theta$

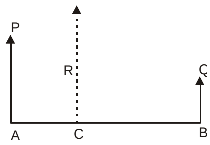
52. **Angular impulse ($\vec{J}$) :**
It is the product of torque and time for which it acts.
Angular impulse = $\vec{J} = \vec{\tau} \times t = I\vec{\alpha}t = I\vec{\omega}_2 - I\vec{\omega}_1 = \vec{L}_2 - \vec{L}_1$
$\vec{J}$ = change in angular momentum

53. **Couple :** Two equal forces with opposite directions, not having the same line of action constitute a couple. e.g. : 1. Turning the cock of a water tap. 2. Turning the key in a lock. 3. Winding a wall clock with a key.

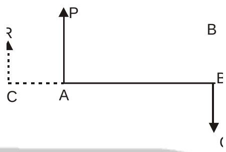
54. The moment of couple or torque is the product of either of the forces and the distance of separation between the forces.

55. To balance a couple, another equal but opposite couple is necessary.

56. Resultant of two like parallel forces :
Resultant (R) = P + Q
P. AC = Q. BC
R lies inside AB, between A and B



57. Resultant of two unlike parallel forces
Resultant (R) = P – Q when P > Q
P. AC = Q. BC
R lies outside AB and is nearer to the greater force



58. **Law of parallel forces :** If a system of coplanar parallel forces acting on a rigid body keep it in equilibrium, then 1) the sum of like parallel forces is equal to the sum of unlike parallel forces or the algebraic sum of the forces is zero and 2) the sum of clockwise moments is equal to the sum of anticlockwise moments or the algebraic sum of the moments at a point is zero. (This is known as principle of law of moments)

59. When the resultant external torque on a system is zero, the angular momentum of the system remains constant. $I_1\omega_1 = I_2\omega_2 = \ldots =$ constant. Circus acrobats, divers and ballet dancers take advantage of this principle.

60. When polar ice cap melts, the duration of the day increases.
(i) If a wheel of radius 'r' rolls on the ground without slipping, the linear velocity of its centre being v, then v = rω.
(ii) The instantaneous velocity of the highest point is 2v and
(iii) The instantaneous velocity of the lowest point is zero.

61. If V is the velocity of centre of mass of a **rolling body** the velocity of **highest point** of rolling body is 2V. Velocity of **lowest point** of the rolling body is **0.** (with respect to an observer out side)

62. The kinetic energy of a rotating body = $\frac{1}{2}I\omega^2 = \frac{1}{2}mv^2\left[\frac{K^2}{R^2}\right]$

63. The total kinetic energy of a rolling body = $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2\left[1+\frac{K^2}{R^2}\right]$

64. Work–energy theorem for a rotating body is given by W = $\frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$

65. Acceleration of a body rolling down an inclined plane without slipping is

$$a = \frac{g\sin\theta}{1+\frac{I}{MR^2}} \text{ or } a = \frac{g\sin\theta}{1+\frac{K^2}{R^2}}$$

66. If a body rolls down an inclined plane of height 'h' without slipping, the velocity acquired is given by

$$V = \sqrt{\frac{2gh}{1+\frac{I}{MR^2}}} \text{ or } V = \sqrt{\frac{2gh}{1+\frac{K^2}{R^2}}}$$

67. The time taken by a body to reach the bottom of an inclined plane of length L and height h is

$$T = \sqrt{\frac{2h(1+\frac{K^2}{R^2})}{g\sin^2\theta}} = \sqrt{\frac{2L(1+\frac{K^2}{R^2})}{g\sin\theta}}$$

68. If a solid sphere, a hollow sphere, a circular disc and a ring are allowed to roll down an inclined plane simultaneously, then the solid sphere reaches the ground first and the ring reaches last.
69. If a_1, a_2, a_3 and a_4 are the accelerations of centre of masses of rolling solid sphere, solid cylinder, hollow sphere and hollow cylinder respectively then.

$$a_1 > a_2 > a_3 > a_4$$

70. If t_1, t_2, t_3 and t_4 are the times of travel of rolling solid sphere, solid cylinder, hollow sphere and hollow cylinder respectively to reach the bottom from the top of an inclined plane then

$$t_1 < t_2 < t_3 < t_4$$

71. The moment of inertia of a body is the least when the axis of rotation passes through the centre of gravity of the body.
72. The acceleration of a rolling body is independent of the mass of the body.
73. A body rolls down an inclined plane without slipping only when the coefficient of friction (μ) bears the relation $\mu \geq \left(\frac{k^2}{k^2+R^2}\right)\tan\theta$. or $\mu \geq \frac{\tan\theta}{\left(1+\frac{k^2}{R^2}\right)}$
74. Formulae for moment of inertia for some important cases :

Object	Axis of rotation	Moment of inertia
1. Disc of radius R	1) through its centre and perpendicular to its plane	$\frac{MR^2}{2}$
	2) about the diameter	$\frac{MR^2}{4}$
	3) about a tangent to its own plane	$\frac{5MR^2}{4}$
	4) tangent perpendicular to the plane of the disc	$\frac{3MR^2}{2}$
2. Annular ring or disc of outer and inner radii R and r	1) through its centre and perpendicular to its plane	$\frac{M(R^2+r^2)}{2}$
	2) about the diameter	$\frac{M(R^2+r^2)}{4}$

	3) about a tangent to its own plane	$\frac{M(5R^2+r^2)}{4}$
3. Solid cylinder of length L and radius R	1) axis of cylinder	$\frac{MR^2}{2}$
	2) through its centre and perpendicular to the axis of cylinder	$M\left(\frac{L^2}{12}+\frac{R^2}{4}\right)$
	3) diameter of the face	$M\left(\frac{L^2}{3}+\frac{R^2}{4}\right)$
4. Thin rod of uniform length L	1) through its centre and perpendicular to its length	$\frac{ML^2}{12}$
	2) through one end and perpendicular to its length	$\frac{ML^2}{3}$
5. Solid sphere of radius R	1) about a diameter	$\frac{2}{5}MR^2$
	2) about a tangent	$\frac{7}{5}MR^2$
6. Hollow sphere of radius R	about a diameter	$\frac{2}{3}MR^2$
7. Thin circular ring of radius R	1) perpendicular to its plane and passing through its centre.	MR^2
	2) about its diameter	$\frac{MR^2}{2}$
8. Hollow cylinder of radius R	about axis of the cylinder	MR^2
9. Rectangular lamina of length l and breadth b	1) through its centre and perpendicular to its plane	$M\left(\frac{l^2}{12}+\frac{b^2}{12}\right)$
	2) through its centre and parallel to breadth along its own plane	$\frac{Ml^2}{12}$
	3) through its centre and parallel to length along its own plane	$\frac{Mb^2}{12}$
	4) edge of the length in the plane of the lamina	$\frac{Mb^2}{3}$
	5) edge of the breadth in the plane of the lamina	$\frac{Ml^2}{3}$
	6) perpendicular to the plane of the lamina and passing through the mid point of the edge of the length	$M\left(\frac{l^2}{12}+\frac{b^2}{3}\right)$
	7) perpendicular to the plane of the lamina and passing through the mid point of the edge of the breadth	$M\left(\frac{l^2}{3}+\frac{b^2}{12}\right)$
10. Plane elliptical lamina with the values of axes 2a and 2b	Perpendicular to the plane of the lamina and passing through its centre	$\frac{M}{4}(a^2+b^2)$

75. Motion along an inclined plane :

Object	Velocity while rolling down an inclined plane	Acceleration = $\dfrac{g\sin\theta}{1+\dfrac{K^2}{R^2}}$
1. Solid sphere	$\sqrt{\dfrac{10gl\sin\theta}{7}}$	$\dfrac{5g\sin\theta}{7}$
2. Hollow sphere	$\sqrt{\dfrac{6gl\sin\theta}{5}}$	$\dfrac{3g\sin\theta}{5}$
3. Solid cylinder	$\sqrt{\dfrac{4gl\sin\theta}{3}}$	$\dfrac{2g\sin\theta}{3}$
4. Disc	$\sqrt{\dfrac{4gl\sin\theta}{3}}$	$\dfrac{2g\sin\theta}{3}$
5. Hollow cylinder	$\sqrt{gl\sin\theta}$	$\dfrac{g\sin\theta}{2}$
6. Ring	$\sqrt{gl\sin\theta}$	$\dfrac{g\sin\theta}{2}$