MATRICES

SYNOPSIS

1. A matrix is an arrangement of real or complex numbers into rows and columns so that all the rows (columns) contain equal number of elements.

2. Two matrices A and B are said to be equal if i) A, B are of same type and ii) the corresponding elements in A and B are equal.

3. $A = [a_{ij}]_{m \times n}$; $B = [b_{jk}]_{n \times p}$ then their product is $[c_{ik}]_{m \times p}$ where $c_{ik} = \sum_{j=1}^{n} a_{ij}.b_{jk}$.

   i) If the product AB exists then it is not necessary that the product BA will also exist.

   ii) Matrix multiplication is not commutative even if AB and BA exist, they need not be equal.

   iii) Matrix multiplication is associative. i.e., A(BC) = (AB)C.

   iv) Let A be a square matrix then $A^2 = A . A$

   $A^2A = A . A^2 = A^3$.

   v) $(A^m)^n = A^{mn}$; $A^m . A^n = A^{m = n}$.

   vi) A(B + C) = AB + AC.

4. A is a matrix of order $m \times n$ then $A.I_n = I_mA = A$

   If A and I are of same order then AI = IA = A

   I is called multiplicative Identity.

5. **Trace of a Matrix**: The sum of the principal diagonal elements a_{11}, a_{22}, a_{33}, ……. a_{nn} of a square matrix $A = [a_{ij}]_{n \times n}$ is called the trace of A. It is denoted by tr A.

   i) tr (KA) = K tr A

   ii) tr (A + B) = tr A + tr B

   iii) tr (A – B) = tr A – tr B.

   iv) tr AB = tr BA

v) tr (AB) ≠ tr (A) . tr (B)

vi) Let A, B, C be three matrices of order n.

Then tr (ABC) = tr (BCA) = tr (CAB) = tr (ACB) = tr (BAC) = tr (CBA)

6. **Transpose of a Matrix**:

$A = [a_{ij}]_{m \times n} \Rightarrow A^T = [a^1_{ji}]_{n \times m}$ where $a^1_{ji} = a_{ij}$.

i) $(A^T)^T = A$

ii) $(A \pm B)^T = A^T \pm B^T$

iii) $(AB)^T = B^T A^T$

iv) $(KA)^T = K.A^T$ (K is a scalar)

7. **Commute**: Two matrices A and B are commute if AB = BA.

8. Let A, B are two square matrices which are commute then

1) $(A + B)^2 = A^2 + 2AB + B^2$

2) $(A - B)^2 = A^2 - 2AB + B^2$.

3) $(A + B)(A - B) = A^2 - B^2$.

4) $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

5) $(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$.

6) $(A + B)(A^2 - AB + B^2) = A^3 + B^3$.

7) $(A - B)(A^2 + AB + B^2) = A^3 - B^3$.

9. If AB = 0 then either A or B need not be equal to 0.

10. If AB = AC then B need not be equal to C even if A ≠ 0.

11. If the elements of a square matrix are polynomials in x and two rows (columns) become identical when x = a then x – 1 is a factor of its determinant.

If three rows are identical then $(x - a)^2$ is a factor.

12. The determinant of a triangular matrix is the product of the elements in the principal diagonal of the matrix.

13. det (AB) = (det A) (det B) = det (BA)

14. If det (AB) = 0 then either det A = 0 or det B = 0.

15. The determinant of a skew symmetric matrix of order 3 is zero.

16. The determinant of a unit matrix is ‘1’.

17. If any row or column of a square matrix contains all its elements as zeros then the determinant of the matrix is 0.

18. A square matrix A is said to be a
    (i) Singular matrix if $|A| = 0$
    (ii) Non singular matrix if $|A| \neq 0$.

19. 1. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$.

2. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$.

3. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

4. $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$

5. $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = (ab+bc+ca)(a-b)(b-c)(c-a)$

6. $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)abc.$

7. $\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix} = 1 + a + b + c.$

8. $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = (a\,b\,c)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

9. $\begin{vmatrix} 1+a^2 & ab & ac \\ ab & 1+b^2 & bc \\ ac & bc & 1+c^2 \end{vmatrix} = 1 + a^2 + b^2 + c^2.$

20. If $A = pa_{ij}]$ is a square matrix of order $n \times n$ and k is a scalar then $|KA| = K^n$ det A.

21. If $A = [a_{ij}]$ is a scalar matrix of order $n \times n$ such that $a_{ij} = k$ for all i then $|A| = K^n$.

22. **Inverse of a Square Matrix:** A square matrix A is said to be an invertible matrix if there exists a square matrix B such that AB = BA = I then B is called the inverse of A.

23. A rectangular matrix cannot be invertible.

24. Every square matrix need not be invertible.

25. An invertible matrix has unique inverse.

26. If A is an invertible matrix then its inverse is denoted by $A^{-1} \Rightarrow A.A^{-1} = A^{-1}.A = I$.

27. If A is invertible $\Rightarrow (A^{-1+)-1} = A_.$

28. If I is invertible matrix $\Rightarrow I^{-1} = I$

29. If A and B are two invertible matrices of same type then AB is also invertible $\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$.

30. $(A_1.A_2 \ldots\ldots A_n)^{-1}.A_n^{-1}.A_{n-1}^{-1} \ldots\ldots A_2^{-1}A_1^{-1}$.

31. If A is an invertible matrix then A^T is also invertible and $(A^t)^{-1} = (A^{-1})^T$.

32. If A is a non singular matrix $\Rightarrow A^{-1} = \frac{\text{Adj A}}{\det A}$

33. If A is a square matrix $\Rightarrow$ A. (Adj A) = (Adj A)A = det A.I.

34. $\det (A^{-1}) = \frac{1}{\det A}$

35. Adj (AB) = (Adj B) (Adj A)

36. If A is a square matrix of type n then $|\text{Adj } A| = |A|n^{n-1}$

37. If A is a non singular matrix of order n, then $\text{Adj } (\text{Adj } A) = |A|^{n-2} A$

38. $(\text{Adj } A)^{-1} = \frac{A}{|A|} = \text{Adj}(A^{-1})$

39. $\text{Adj } A^T = (\text{Adj } A)^T$.

40. For any scale k, $\text{Adj } (kA) = kn^{-1} \text{ Adj } A$.

41. $|\text{Adj } (\text{Adj } A)| = |A|^{(n-1)^2}$.

42. $|\text{Adj Adi Adi } A| = |A|^{(n-1)^3}$

43. If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$.

44. If A = $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ and abc ≠ 0 then $A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$.

45. If A(α) = $\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $[A(\alpha)^{-1} = A(-\alpha)$.

46. The inverse of a symmetric matrix is symmetric.

47. The inverse of a diagonal matrix is a diagonal matrix.

48. A is singular $\Rightarrow A^T$ is singular.

A is non singular $\Rightarrow A^T$ is non singular.

49. If A and B are non singular matrices of the same type then AB is a non singular of the same type.

50. If A is a singular matrix then Adj A is also singular matrix.

51. If A is a singular then A (Adj A) = (Adj A)A = 0.

52. If |A| = 0, then |Adj A| = 0.

53. If A is symmetric then Adj A is also symmetric.

54. i) The linear equations in two variable are $a_1x + b_1y = c_1$.

$a_2x + b_2y = c_2$ then the system of equations in x and y can be written as the matrix equation AX = B.

Where A = $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ x = $\begin{bmatrix} x \\ y \end{bmatrix}$, B = $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

ii) The Homogeneous Equation are $a_1x + b_1y = 0$; $a_2x + b_2y = 0$.

If x = 0, y = 0 then the solution is called **zero solution** (Trivial solution). $|A| \neq 0$ other wise the solution is called **non trivial solution,** $|A| = 0$.

iii) The linear equations in three variables are

$a_1x + b_1y + c_1z = d_1$;

$a_2x + b_2y + c_2z = d_2$;

$a_3x + b_3y + c_3z = d_3$

Matrix equation is AX = D where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \; D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

iv) The homogeneous equations are

$a_1x + b_1y + c_1z = 0$;

$a_2x + b_2y + c_2z = 0$;

$a_3x + b_3y + c_3z = 0$

If x = 0, y = 0, z = 0 then the solution is called **zero solution** (Trivial solution)

Other wise the solution is called **non trivial solution**.

If the system of equation is AX = 0 where A is non singular, then the system possess trivial solution only.

If the system of equation is AX = 0; where A is singular, then the system possesses a non trivial solution.

v) The system of equation AX = B or AX = D is said to be **consistent** if AX = B or AX = D has a solution.

vi) The system of equations AX = B or AX = D is said to the **inconsistent** if AX = B or AX = D has no solution.

I. Consider the system of equation are

$a_1x + b_1y + c_1z = d_1$;

$a_2x + b_2y + c_2z = d_2$;

$a_3x + b_3y + c_zz = d_3$

i) The matrix $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is called **coefficient matrix**.

ii) The matrix $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$ is called as **Augmented matrix**.

iii) The Augmented Matrix can be reduced into the form $\begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \end{bmatrix}$ then $x = \alpha$, $y = \beta$, $z = \gamma$ is the solution of the system of equations.

iv) The Augmented matrix is reduced to the form $\begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ 0 & p_5 & p_6 & p_7 \\ 0 & 0 & p_8 & p_9 \end{bmatrix}$ by using elementary trans-formations known as **Echelon** form of a matrix.

II. Sub Matrix: A matrix obtained by deleting some rows or columns (or both) of a matrix is called as sub matrix.

i) Every element of matrix is a sub matrix of order 1.

ii) Every matrix is a sub matrix of itself.

III. Rank of a Matrix: The rank of matrix is the order of the highest order non singular square matrix.

We can find the rank of a matrix by reducing the system of equations into Echelon form.

i) The rank of a matrix in Echelon form is equal to the number of non zero rows of the matrix.

ii) The rank of a unit matrix of order n is n.

iii) The rank of a non singular matrix of order n is n.

IV. Let AX = B be a system of equations in a unknowns, such that the rank of the coefficient matrix A is r_1 and the rank of the augmented matrix K is r_2.

i) If $r_1 \neq r_2$ then the system AX = B is inconsistent i.e. it has no solution.

ii) If $r_1 = r_2 = n$ then the system AX = B is consistent and it has unique solution.

iii) If $r_1 = r_2 < n$ then the system AX = B is consistent and it has infinitely many solutions.