MATHEMATICAL METHODS

INTERPOLATION

I YEAR B.Tech

By

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# SYLLABUS OF MATHEMATICAL METHODS (as per JNTU Hyderabad)

<table>
<thead>
<tr>
<th>Name of the Unit</th>
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<tr>
<td><strong>Unit-II</strong>&lt;br&gt;Eigen values and Eigen vectors</td>
<td>Eigen values, Eigen vectors – properties – Condition number of Matrix, Cayley – Hamilton Theorem (without proof) – Inverse and powers of a matrix by Cayley – Hamilton theorem – Diagonalization of matrix – Calculation of powers of matrix – Model and spectral matrices.</td>
</tr>
<tr>
<td><strong>Unit-V</strong>&lt;br&gt;Curve fitting &amp; Numerical Integration</td>
<td><strong>Curve Fitting</strong>: Fitting a straight line - Second degree curve - Exponential curve - Power curve by method of least squares. <strong>Numerical Integration</strong>: Numerical Differentiation-Simpson’s 3/8 Rule, Gaussian Integration, Evaluation of Principal value integrals, Generalized Quadrature.</td>
</tr>
<tr>
<td><strong>Unit-VI</strong>&lt;br&gt;Numerical solution of ODE</td>
<td>Solution by Taylor's series - Picard's Method of successive approximation- Euler's Method -Runge kutta Methods, Predictor Corrector Methods, Adams- Bashforth Method.</td>
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<tr>
<td><strong>Unit-VII</strong>&lt;br&gt;Fourier Series</td>
<td>Determination of Fourier coefficients - Fourier series-even and odd functions - Fourier series in an arbitrary interval - Even and odd periodic continuation - Half-range Fourier sine and cosine expansions.</td>
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<td><strong>Unit-VIII</strong>&lt;br&gt;Partial Differential Equations</td>
<td>Introduction and formation of PDE by elimination of arbitrary constants and arbitrary functions - Solutions of first order linear equation - Non linear equations - Method of separation of variables for second order equations - Two dimensional wave equation.</td>
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UNIT-IV(b)
INTERPOLATION

- Introduction
- Introduction to Forward, Backward and Central differences
- Symbolic relations and Separation of Symbols
- Properties
- Newton's Forward Difference Interpolation Formulae
- Newton's Backward Difference Interpolation Formulae
- Gauss Forward Central Difference Interpolation Formulae
- Gauss Backward Central Difference Interpolation Formulae
- Striling's Formulae
- Lagrange's Interpolation
INTERPOLATION

The process of finding the curve passing through the points \((x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) is called as Interpolation and the curve obtained is called as Interpolating curve.

Interpolating polynomial passing through the given set of points is unique.

Let \(x_0, x_1, x_2, \ldots, x_n\) be given set of observations and \(y = f(x)\) be the given function, then the method to find \(f(x_m) \forall x_0 \leq x_m \leq x_n\) is called as an Interpolation.

If \(x_m\) is not in the range of \(x_0\) and \(x_n\), then the method to find \((x_m)\) is called as Extrapolation.

\[
\begin{array}{c}
 x_0, x_1, x_2, \ldots, x_n \\
\text{Equally Spaced} \\
\text{Arguments} \\
\text{Unequally Spaced} \\
\text{Arguments} \\
\text{Newton's & Gauss} \\
\text{Interpolation} \\
\text{Lagranges} \\
\text{Interpolation}
\end{array}
\]

The Interpolation depends upon finite difference concept.

If \(x_0, x_1, x_2, \ldots, x_n\) be given set of observations and let \(y_0 = f(x_0), y_1 = f(x_1), \ldots, y_n = f(x_n)\) be their corresponding values for the curve \(y = f(x)\), then \(y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}\) is called as finite difference.

When the arguments are equally spaced i.e. \(x_i - x_{i-1} = h \forall i\) then we can use one of the following differences.

- Forward differences
- Backward differences
- Central differences

Forward Difference

Let us consider \(x_0, x_1, x_2, \ldots, x_n\) be given set of observations and let \(y_0, y_1, y_2, \ldots, y_n\) are corresponding values of the curve \(y = f(x)\), then the Forward difference operator is denoted by \(\Delta\) and is defined as \(\Delta y_i = y_i - y_{i-1}\), \(\Delta y_1 = y_2 - y_1, \ldots, \Delta y_{n-1} = y_n - y_{n-1}\).

In this case \(\Delta y_0, \Delta y_1, \ldots, \Delta y_n\) are called as First Forward differences of \(y\).

The difference of first forward differences will give us Second forward differences and it is denoted by \(\Delta^2\) and is defined as \(\Delta^2 y_0 = \Delta(\Delta y_0)\)

\[
= \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 \\
= (y_2 - y_1) - (y_1 - y_0) \\
= y_2 - 2y_1 - y_0
\]
Similarly, the difference of second forward differences will give us third forward difference and it is denoted by $\Delta^3$.

**Forward difference table**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = f(x)$</th>
<th>First Forward differences $\Delta y$</th>
<th>Second Forward differences $\Delta^2 y$</th>
<th>Third Forward differences $\Delta^3 y$</th>
<th>Fourth differences $\Delta^4 y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0$</td>
<td>$y_0$</td>
<td>$\Delta y_0 = y_1 - y_0$</td>
<td>$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$</td>
<td>$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_2$</td>
<td>$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>$y_1$</td>
<td>$\Delta y_1 = y_2 - y_1$</td>
<td></td>
<td>$\Delta^3 y_1 = \Delta^2 y_3 - \Delta^2 y_2$</td>
<td></td>
</tr>
<tr>
<td>$x_2$</td>
<td>$y_2$</td>
<td></td>
<td>$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_3$</td>
<td>$y_3$</td>
<td>$\Delta y_2 = y_3 - y_2$</td>
<td>$\Delta^3 y_1 = \Delta^2 y_3 - \Delta^2 y_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$x_{n-1}$</td>
<td>$y_{n-1}$</td>
<td>$\Delta y_{n-1} = \Delta y_n - \Delta y_{n-1}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_n$</td>
<td>$y_n$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Note:** If $h$ is common difference in the values of $x$ and $y = f(x)$ be the given function then $\Delta f(x) = f(x + h) - f(x)$.

**Backward Difference**

Let us consider $x_0, x_1, x_2, ..., x_n$ be given set of observations and let $y_0, y_1, y_2, ..., y_n$ are corresponding values of the curve $y = f(x)$, then the Backward difference operator is denoted by $\nabla$ and is defined as $\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1$, ..., $\nabla y_n = y_n - y_{n-1}$.

In this case $\nabla y_0, \nabla y_1, ..., \nabla y_n$ are called as First Backward differences of $y$.

The difference of first Backward differences will give us Second Backward differences and it is denoted by $\nabla^2$ and is defined as $\nabla^2 y_2 = \nabla (\nabla y_2)$

$$\nabla (y_2 - y_1) = \nabla y_2 - \nabla y_1$$

$$= (y_2 - y_1) - (y_1 - y_0)$$

$$= y_2 - 2y_1 - y_0$$

Similarly, the difference of second backward differences will give us third backward difference and it is denoted by $\nabla^3$. 
Backward difference table

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = f(x)$</th>
<th>First Backward differences $\Delta y$</th>
<th>Second Backward differences $\Delta^2 y$</th>
<th>Third Backward differences $\Delta^3 y$</th>
<th>Fourth differences $\Delta^4 y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0$</td>
<td>$y_0$</td>
<td>$\nabla y_1 = y_1 - y_0$</td>
<td>$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>$y_1$</td>
<td>$\nabla y_2 = y_2 - y_1$</td>
<td>$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$</td>
<td>$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$</td>
<td></td>
</tr>
<tr>
<td>$x_2$</td>
<td>$y_2$</td>
<td>$\nabla y_3 = y_3 - y_2$</td>
<td>$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_3$</td>
<td>$y_3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_{n-1}$</td>
<td>$y_{n-1}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_n$</td>
<td>$y_n$</td>
<td>$\nabla y_n = \nabla y_n - \nabla y_{n-1}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Note:** If $h$ is common difference in the values of $x$ and $y = f(x)$ be the given function then $\nabla f(x + h) = f(x + h) - f(x)$.

### Central differences

Let us consider $x_0, x_1, x_2, \ldots, x_n$ be a given set of observations and let $y_0, y_1, y_2, \ldots, y_n$ are corresponding values of the curve $y = f(x)$, then the Central difference operator is denoted by $\delta$ and is defined as:

- If $n$ is odd: $\delta^n y_{r-\frac{1}{2}} = \delta^{n-1} y_r - \delta^{n-1} y_{r-1}$, $r = 1, 2, 3, \ldots$
- If $n$ is even: $\delta^n y_r = \delta^{n-1} y_{r+\frac{1}{2}} - \delta^{n-1} y_{r-\frac{1}{2}}$, $r = 1, 2, 3, \ldots$

and $\delta^0 y_r = y_r$

The Central difference table is shown below

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$\delta y$</th>
<th>$\delta^2 y$</th>
<th>$\delta^3 y$</th>
<th>$\delta^4 y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0$</td>
<td>$y_0$</td>
<td>$\delta y_{\frac{1}{2}}$</td>
<td>$\delta^2 y_1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>$y_1$</td>
<td>$\delta y_{\frac{3}{2}}$</td>
<td>$\delta^2 y_2$</td>
<td>$\delta^3 y_{\frac{3}{2}}$</td>
<td></td>
</tr>
<tr>
<td>$x_2$</td>
<td>$y_2$</td>
<td>$\delta y_{\frac{5}{2}}$</td>
<td>$\delta^2 y_3$</td>
<td></td>
<td>$\delta^4 y_2$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$y_3$</td>
<td>$\delta y_{\frac{7}{2}}$</td>
<td></td>
<td>$\delta^3 y_{\frac{7}{2}}$</td>
<td></td>
</tr>
<tr>
<td>$x_4$</td>
<td>$y_4$</td>
<td></td>
<td>$\delta^2 y_3$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Note:** Let $h$ be common difference in the values of $x$ and $y = f(x)$ be the given function then $\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$
Symbolic Relations and Separation of Symbols

**Average Operator:** The average operator \( \mu \) is defined by the equation
\[
\mu y_r = \frac{1}{2} \left( y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}} \right)
\]
(Or)

Let \( h \) is the common difference in the values of \( x \) and \( y = f(x) \) be the given function, then the average operator is denoted by \( \mu \) and is defined as \( \mu f(x) = \frac{f(x+h) + f(x-h)}{2} \)

**Shift Operator:** The Shift operator \( E \) is defined by the equation \( Ey_r = y_{r+1} \)

Similarly, \( E^n y_r = y_{n+r} \)

(Or)

Let \( h \) is the common difference in the values of \( x \) and \( y = f(x) \) be the given function, then the shift operator is denoted by \( E \) and is defined as \( Ef(x) = f(x + h) \)

**Inverse Operator:** The Inverse Operator \( E^{-1} \) is defined as \( E^{-1}y_r = y_{r-1} \)

In general, \( E^{-n} y_r = y_{r-n} \)

**Properties**

1) **Prove that** \( E = 1 + \Delta \)

**Sol:** Consider R.H.S: \((1 + \Delta)y_n = y_n + \Delta y_n\)
\[
= y_n + (y_{n+1} - y_n)
= y_{n+1}
= E^1 y_n \quad (\therefore E^n y_r = y_{n+r})
\]
\[
\therefore E = 1 + \Delta
\]

2) **Prove that** \( \nabla = 1 - E^{-1} \)

**Sol:** Consider L.H.S: \( \nabla y_n = y_n - y_{n-1} \)
\[
= y_n - E^{-1} y_n
= (1 - E^{-1}) y_n
\]
\[
\therefore \nabla = 1 - E^{-1}
\]

3) **Prove that** \( \Delta = E \nabla = \nabla E \)

**Sol:** **Case (i)** Consider \((E \nabla)y_n = E(\nabla y_n)\)
\[
= E(y_n - y_{n-1})
= E y_n - E y_{n-1}
= y_{n+1} - y_n
= \Delta y_n
\]
\[
\therefore \Delta = E \nabla
\]

**Case (ii)** Consider \((\nabla E)y_n = \nabla(E y_n)\)
\[
= \nabla(y_{n+1})
= y_{n+1} - y_n
= \Delta y_n
\]
\[
\therefore \Delta = \nabla E
\]

Hence from these cases, we can conclude that \( \Delta = E \nabla = \nabla E \)

4) **Prove that** \((1 + \Delta)(1 - \nabla) = 1 \)

**Sol:** Consider \((1 + \Delta)(1 - \nabla)y_n = (1 + \Delta)(y_n - \nabla y_n)\)
\[
= (1 + \Delta)(y_n - \{y_n - y_{n-1}\})
= (1 + \Delta)y_{n-1}
\]
\[ \Delta = \nabla (1 - \nabla)^{-1} \]

5) Prove that \( \Delta = \nabla (1 - \nabla)^{-1} \) (Hint: Consider \( \Delta (1 - \nabla) \))

6) Prove that \( (1 + \Delta) = (E - 1) \nabla^{-1} \)

7) Prove that \( \delta = E^{1/2} - E^{-1/2} \)

Sol: We know that \( \delta y_r = y_{r+1/2} - y_{r-1/2} \)

\[ = \left( E^{1/2} - E^{-1/2} \right) y_r \]

Hence the result \( \delta = E^{1/2} - E^{-1/2} \)

8) Prove that \( \mu \equiv \frac{1}{2} \left( E^{1/2} - E^{-1/2} \right) \)

Sol: We know that \( \mu y_r = \frac{1}{2} \left( y_{r+1/2} + y_{r-1/2} \right) \)

\[ = \frac{1}{2} \left( E^{1/2} + E^{-1/2} \right) y_r \]

Hence proved that \( \mu \equiv \frac{1}{2} \left( E^{1/2} - E^{-1/2} \right) \)

9) Prove that \( \mu^2 = 1 + \frac{1}{4} \delta^2 \)

Sol: We know that \( \delta = E^{1/2} - E^{-1/2} \)

Squaring on both sides, we get \( \delta^2 = \left( E^{1/2} - E^{-1/2} \right)^2 \)

L.H.S \[ \Rightarrow 1 + \frac{1}{4} \delta^2 = 1 + \frac{1}{4} (E^1 + E^{-1} - 2) \]

\[ = \frac{1}{4} (E^1 + E^{-1} + 2) = \mu^2 \]

Hence the result

Relation between the operator \( D \) and \( E \)

Here Operator \( D \equiv \frac{d}{dx} \)

We know that \( Ef(x) = f(x + h) \)

Expanding using Taylor’s series, we get

\[ Ef(x) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \ldots \]

\[ = [1 + hD + h^2 D^2 + \ldots ] f(x) \]

\[ = e^{hD} f(x) \]

\[ \Rightarrow E = e^{hD} \]
Newton’s Forward Interpolation Formula

Statement: If \( x_0, x_1, x_2, ..., x_n \) are given set of observations with common difference \( h \) and let \( y_0, y_1, y_2, ..., y_n \) are their corresponding values, where \( y = f(x) \) be the given function then

\[
f(x) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \cdots + \frac{p(p-1)(p-2)...(p-(n-1))}{n!} \Delta^n y_0
\]

where \( p = \frac{x-x_0}{h} \)

Proof: Let us assume an \( n^{th} \) degree polynomial

\[
f(x) = A_0 + A_1 (x - x_0) + A_2 (x - x_0)(x - x_1) + \cdots + A_n (x - x_0)(x - x_1) \cdots (x - x_{n-1}) --- (i)
\]

Substitute \( x = x_0 \) in (i), we get \( f(x_0) = A_0 \Rightarrow y_0 = A_0 \)

Substitute \( x = x_1 \) in (i), we get \( f(x_1) = A_0 + A_1 (x_1 - x_0) \Rightarrow y_1 = y_0 + A_1 h \)

\[
\Rightarrow A_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}
\]

Substitute \( x = x_2 \) in (i), we get \( f(x_2) = A_0 + A_1 (x_2 - x_0) + A_2 (x_2 - x_0)(x_2 - x_1) \)

\[
\Rightarrow y_2 = y_0 + A_1 (2h) + A_2 (2h)(h)
\]

\[
\Rightarrow y_2 = y_0 + 2h \left( \frac{\Delta y_0}{h} \right) + 2h^2 A_2
\]

\[
\Rightarrow A_2 = \frac{1}{2h^2} \Delta^2 y_0
\]

Similarly, we get \( A_n = \frac{1}{n!h^2} \Delta^n y_0 \)

Substituting these values in (i), we get

\[
f(x) = y_0 + (x-x_0) \frac{1}{h} \Delta y_0 + (x-x_0)(x-x_1) \frac{1}{2h^2} \Delta^2 y_0 + \cdots + (x-x_0)(x-x_1) \cdots (x-x_{n-1}) \frac{1}{n!h^2} \Delta^n y_0
\]

---(ii)

But given \( p = \frac{x-x_0}{h} \)

\[
\Rightarrow x - x_0 = ph \Rightarrow x = x_0 + h
\]

\[
\Rightarrow x - x_1 = x - (x_0 + h)
\]

\[
= (x - x_0) - h
\]

\[
= ph - h = (p - 1)h
\]

Similarly, \( x - x_2 = (p - 2)h \),

\[
\vdots
\]

\[
x - x_{n-1} = (p - (n - 1))h
\]

Substituting in the Equation (ii), we get

\[
f(x) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \cdots + \frac{p(p-1)(p-2)...(p-(n-1))}{n!} \Delta^n y_0
\]
Newton's Backward Interpolation Formula

Statement: If \( x_0, x_1, x_2, ..., x_n \) are given set of observations with common difference \( h \) and let 
\( y_0, y_1, y_2, ..., y_n \) are their corresponding values, where \( y = f(x) \) be the given function then

\[
f(x) = y_n + p \frac{p(p+1)}{2!} \Delta^2 y_n + p \frac{(p+1)(p+2)}{3!} \Delta^3 y_n + ... + p \frac{(p+1)(p+2)...(p+(n-1))}{n!} \Delta^n y_n
\]

where \( p = \frac{x-x_0}{h} \)

Proof: Let us assume an \( n^{th} \) degree polynomial

\[
f(x) = A_0 + A_1(x - x_n) + A_2(x - x_n)(x - x_{n-1}) + ... + A_n (x - x_n)(x - x_{n-1}) ...(x - x_i)
\]

\[\Rightarrow (i)\]

Substitute \( x = x_n \) in (i), we get \( f(x_n) = A_0 \Rightarrow y_n = A_0 \)

Substitute \( x = x_{n-1} \) in (i), we get \( f(x_{n-1}) = A_0 + A_1(x_{n-1} - x_n) \Rightarrow y_{n-1} = y_n - A_1 h \)

\[\Rightarrow A_1 = \frac{y_n-y_{n-1}}{h} = \frac{\Delta y_n}{h}\]

Substitute \( x = x_{n-2} \) in (i), we get \( f(x_{n-2}) = A_0 + A_1(x_{n-2} - x_n) + A_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1}) \)

\[\Rightarrow y_{n-2} = y_n + A_1(-2h) + A_2(-2h)(-h) \]

\[\Rightarrow y_{n-2} = y_n - 2h \left( \frac{\Delta^2 y_n}{h} \right) + 2h^2 A_2 \]

\[\Rightarrow A_2 = \frac{1}{2h^2} \Delta^2 y_n \]

Similarly, we get \( A_n = \frac{1}{n!h^n} \Delta^n y_n \)

Substituting these values in (i), we get

\[
f(x) = y_n + (x-x_n) \frac{1}{h} \Delta y_n + (x-x_n)(x-x_{n-1}) \frac{1}{2h^2} \Delta^2 y_n + ...
\]

\[+(x-x_n)(x-x_{n-1}) ...(x-x_1) \frac{1}{nh^2} \Delta^n y_n \quad \text{-----(ii)} \]

But given \( p = \frac{x-x_n}{h} \)

\[\Rightarrow x - x_n = ph \Rightarrow x = x_n + h \]

\[\Rightarrow x - x_{n-1} = x - (x_n - h) \]

\[= (x - x_n) + h \]

\[= ph + h = (p + 1)h \]

Similarly, \( x - x_{n-2} = (p + 2)h \),

\[\vdots\]

\[x - x_1 = (p + (n - 1))h \]

Substituting in the Equation (ii), we get

\[
f(x) = y_n + p \frac{p(p+1)}{2!} \Delta^2 y_n + p \frac{(p+1)(p+2)}{3!} \Delta^3 y_n + ... + p \frac{(p+1)(p+2)...(p+(n-1))}{n!} \Delta^n y_n
\]
Gauss forward central difference formula

Statement: If ..., \(x_{-2}, x_{-1}, x_0, x_1, x_2,\ldots\) are given set of observations with common difference \(h\) and let ..., \(y_{-2}, y_{-1}, y_0, y_1, y_2,\ldots\) are their corresponding values, where \(y = f(x)\) be the given function then \(y_p = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!} \Delta^3 y_{-2} + \frac{p(p-1)(p+1)(p-2)}{4!} \Delta^4 y_{-2} + \ldots\)

where \(p = \frac{x-x_0}{h}.

Proof:

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
x & y & \Delta y & \Delta^2 y & \Delta^3 y & \Delta^4 y \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_{-2} & y_{-2} & \Delta y_{-2} & \Delta^2 y_{-2} & \Delta^3 y_{-2} & \Delta^4 y_{-2} \\
x_{-1} & y_{-1} & \Delta y_{-1} & \Delta^2 y_{-1} & \Delta^3 y_{-1} & \Delta^4 y_{-2} \\
x_0 & y_0 & \Delta y_0 & \Delta^2 y_0 & \Delta^3 y_0 & \Delta^4 y \\
x_1 & y_1 & \Delta y_0 & \Delta^2 y_0 & \Delta^3 y_0 & \Delta^4 y \\
x_2 & y_2 & \Delta y_{-1} & \Delta^2 y_{-1} & \Delta^3 y_{-1} & \Delta^4 y_{-2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
\end{array}
\]

Let us assume a polynomial equation by using the arrow marks shown in the above table.

Let \(y_p = y_0 + G_1 \Delta y_0 + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-1} + G_4 \Delta^4 y_{-2} + \ldots ---- (1)\)

where \(G_0, G_1, G_2, \ldots\) are unknowns

\(y_p = y_{p+0} = E^p y_0 = (1 + \Delta)^p y_0 \quad (\because E = 1 + \Delta)\)

\[\Rightarrow y_p = \left(1 + p c_1 \Delta + p c_2 \Delta^2 + p c_3 \Delta^3 + \ldots + p c_p \Delta^p\right) y_0\]

\[\Rightarrow y_p = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \ldots --- (2)\]

Now, \(y_{-1} = y_{-1+0} = E^{-1} y_0 = (1 + \Delta)^{-1} y_0\)

\[= (1 - \Delta + \Delta^2 - \Delta^3 + \ldots) y_0\]

\[\Rightarrow y_{-1} = y_0 - \Delta y_0 + \Delta^2 y_0 - \ldots\]

Therefore, \(\Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_0 + \ldots ---- (3)\)

and \(\Delta^3 y_{-1} = \Delta^3 y_0 - \Delta^4 y_0 + \ldots ---- (4)\)

Substituting 2, 3, 4 in 1, we get

\[y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \ldots = y_0 + G_1 \Delta y_0 + G_2 (\Delta^2 y_0 - \Delta^3 y_0 + \ldots) + \]

\[G_3 (\Delta^3 y_0 - \Delta^4 y_0 + \ldots) + \ldots\]

Comparing corresponding coefficients, we get
Similarly, \( G_4 = \frac{p(p+1)(p+2)}{4!} \)

Substituting all these values of \( G_0, G_1, G_2, \ldots \) in (1), we get

\[
y_p = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!} \Delta^3 y_{-1} + \frac{p(p-1)(p+1)(p+2)}{4!} \Delta^4 y_{-2} + \ldots
\]

**Gauss backward central difference formula**

**Statement:** If \( ..., x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots \) are given set of observations with common difference \( h \) and let \( ..., y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots \) are their corresponding values, where \( y = f(x) \) be the given function then

\[
y_p = y_0 + p \Delta y_{-1} + \frac{p(p+1)}{2!} \Delta^2 y_{-1} + \frac{p(p+1)(p+1)}{3!} \Delta^3 y_{-2} + \frac{p(p+1)(p+1)(p+2)}{4!} \Delta^4 y_{-2} + \ldots
\]

where \( p = \frac{x-x_0}{h} \).

**Proof:**

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<tr>
<th>( x )</th>
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Let us assume a polynomial equation by using the arrow marks shown in the above table.

Let \( y_p = y_0 + G_1 \Delta y_{-1} + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-2} + G_4 \Delta^4 y_{-2} + \ldots \quad (1) \)

where \( G_0, G_1, G_2, \ldots \) are unknowns

\[
y_p = y_{p+0} = E^p y_0 = (1+\Delta)^p y_0 \quad (\because E = 1+\Delta)
\]

\[
\Rightarrow y_p = \left(1 + p c_1 \Delta + p c_2 \Delta^2 + p c_3 \Delta^3 + \ldots + p c_p \Delta^p\right) y_0
\]

\[
\Rightarrow y_p = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p+1)}{3!} \Delta^3 y_0 + \ldots \quad (2)
\]

Now, \( y_{-1} = y_{-1+0} = E^{-1} y_0 = (1+\Delta)^{-1} y_0 \)

\[
= (1 - \Delta + \Delta^2 - \Delta^3 + \ldots) y_0
\]

\[
\Rightarrow y_{-1} = y_0 - \Delta y_0 + \Delta^2 y_0 - \ldots
\]
Therefore, \( \Delta y_{-1} = \Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 - \ldots \) ---- (3)

\[ \Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_0 + \ldots \] ---- (4)

Also \( y_{-2} = y_{-2+0} = E^{-2} y_0 = (1 + \Delta)^{-2} y_0 \)

\[ = (1 - 2\Delta + 3\Delta^2 - 4\Delta^3 + \ldots ) y_0 \]

\[ \implies y_{-2} = y_0 - 2\Delta y_0 + 3\Delta^2 y_0 - \ldots \]

Now, \( \Delta^3 y_{-2} = \Delta^3 y_0 - 2\Delta^4 y_0 + \ldots \) ---- (5)

Substituting 2, 3, 4, 5 in 1, we get

\[ y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 \ldots = y_0 + G_1 (\Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 - \ldots ) + G_2 (\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \ldots ) + G_3 (\Delta^3 y_0 - 2\Delta^4 y_0 + \ldots ) + \ldots \]

Comparing corresponding coefficients, we get

\[ G_1 = p, \quad -G_1 + G_2 = \frac{p(p-1)}{2!} \quad \implies \quad G_2 = \frac{p(p+1)}{2!} \]

Also, \( G_1 - G_2 + G_3 = \frac{p(p-1)(p-2)}{3!} \quad \implies \quad G_3 = \frac{p(p+1)(p-1)}{3!} \)

Similarly, \( G_4 = \frac{p(p+1)(p-1)(p+2)}{4!}, \ldots \)

Substituting all these values of \( G_0, G_1, G_2, \ldots \) in (1), we get

\[ y_p = y_0 + p \Delta y_1 + \frac{p(p+1)}{2!} \Delta^2 y_1 + \frac{p(p+1)(p-1)}{3!} \Delta^3 y_1 + \frac{p^2(p-1)}{4!} \Delta^4 y_1 + \ldots, \quad p = \frac{x-x_0}{h} \]

**Stirling’s Formulae**

**Statement:** If \( ..., x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots \) are given set of observations with common difference \( h \) and \( ..., y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots \) are their corresponding values, where \( y = f(x) \) be the given function then

\[ y_p = y_0 + p \left( \frac{\Delta y_0 + \Delta y_1}{2} \right) + \frac{p^2}{2!} \Delta^2 y_1 + \frac{p(p+1)}{3!} \left( \frac{\Delta^2 y_1 + \Delta^3 y_2}{2} \right) + \frac{p^2(p-1)}{4!} \Delta^4 y_2 + \ldots \]

where \( p = \frac{x-x_0}{h} \)

**Proof:** Stirling’s Formula will be obtained by taking the average of Gauss forward difference formula and Gauss Backward difference formula.

We know that, from Gauss forward difference formula

\[ y_p = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_1 + \frac{p(p-1)(p+1)}{3!} \Delta^3 y_1 + \frac{p(p-1)(p+1)(p-2)}{4!} \Delta^4 y_2 + \ldots \] ---- (1)

Also, from Gauss backward difference formula

\[ y_p = y_0 + p \Delta y_{-1} + \frac{p(p+1)}{2!} \Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!} \Delta^3 y_{-1} + \frac{p(p+1)(p-1)(p+2)}{4!} \Delta^4 y_{-2} + \ldots \] ---- (2)

Now, **Stirling’s Formula** \( = \frac{1}{2} \) (Gauss forward formula + Gauss backward formula)

\[ \therefore y_p = y_0 + p \left( \frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2 - 1)}{3!} \left( \frac{\Delta^2 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \frac{p^2(p^2 - 1)}{4!} \Delta^4 y_{-2} + \ldots \]
Lagrange’s Interpolation Formula

**Statement:** If \( x_0, x_1, x_2, ..., x_n \) are given set of observations which are need not be equally spaced and let \( y_0, y_1, y_2, ..., y_n \) are their corresponding values, where \( y = f(x) \) be the given function then

\[
f(x) = \frac{(x-x_1)(x-x_2) ... (x-x_n)}{(x_0-x_1)(x_0-x_2) ... (x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2) ... (x-x_n)}{(x_1-x_0)(x_1-x_2) ... (x_1-x_n)} y_1 + \cdots + \frac{(x-x_0)(x-x_1) ... (x-x_{n-1})}{(x_n-x_0)(x_n-x_1) ... (x_n-x_{n-1})} y_n
\]

**Proof:** Let us assume an \( n^{th} \) degree polynomial of the form

\[
f(x) = A_0(x-x_1)(x-x_2) ... (x-x_n) + A_1(x-x_0)(x-x_2) ... (x-x_n) + \cdots + A_n(x-x_0)(x-x_1) ... (x-x_{n-1})
\]

Substitute \( x = x_0 \) , we get

\[
f(x_0) = A_0(x_0-x_1)(x_0-x_2) ... (x_0-x_n)
\]

\[
\Rightarrow y_0 = A_0(x_0-x_1)(x_0-x_2) ... (x_0-x_n)
\]

\[
\Rightarrow A_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2) ... (x_0-x_n)}
\]

Again, \( x = x_1 \), we get

\[
f(x_1) = A_1(x_1-x_0)(x_1-x_2) ... (x_1-x_n)
\]

\[
\Rightarrow y_1 = A_1(x_1-x_0)(x_1-x_2) ... (x_1-x_n)
\]

\[
\Rightarrow A_1 = \frac{y_1}{(x_1-x_0)(x_1-x_2) ... (x_1-x_n)}
\]

Proceeding like this, finally we get

\[
A_n = \frac{y_n}{(x_n-x_0)(x_n-x_1)(x_n-x_{n-1})}
\]

Substituting these values in the Equation (1), we get

\[
f(x) = \frac{(x-x_1)(x-x_2) ... (x-x_n)}{(x_0-x_1)(x_0-x_2) ... (x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2) ... (x-x_n)}{(x_1-x_0)(x_1-x_2) ... (x_1-x_n)} y_1 + \cdots + \frac{(x-x_0)(x-x_1) ... (x-x_{n-1})}{(x_n-x_0)(x_n-x_1) ... (x_n-x_{n-1})} y_n
\]

**Note:** This Lagrange’s formula is used for both equally spaced and unequally spaced arguments.