PARTIAL FRACTIONS

I. Remainder obtained when f(x) is divided by x – a is f(a).

If degree of divisor is 'n', then the degree of remainder is (n – 1).

f(x), g(x) are two polynomials. If $g(x) \neq 0$, then $\exists$ two polynomials q(x) , r(x) such that

$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$ if the degree of f(x) is > that of g(x).

II. Method of resolving proper fraction $\frac{f(x)}{g(x)}$ into partial fractions.

Type 1: When the denominator g(x) contains non-repeated linear factors i.e.

g(x) = (x – a) (x – b) (x – c).

$$\frac{f(x)}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

Type 2: When the denominator g(x) contains repeated and non repeated linear factors.

i.e. $g(x) = (x – a)^2 (x – b)$,

$$\frac{f(x)}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$$

Type 3: When the denominator g(x) contains non repeated irreducible quadratic factors.

i.e. $g(x) = (ax^2 + bx + c)(x – d)$.

$$\frac{f(x)}{(ax^2+bx+c)(x-d)} = \frac{Ax+B}{(ax^2+bx+c)} + \frac{C}{x-d}.$$

Type 4: When the denominator g(x) contains repeated irreducible quadratic factors

i.e. $g(x) = (ax^2 + bx + c)^2 (x – d)$

$$\frac{f(x)}{(ax^2+bx+c)^2(x-d)} = \frac{Ax+B}{(ax^2+bx+c)} + \frac{Cx+D}{(ax^2+bx+c)^2} + \frac{E}{x-d}$$

Very Short Answer Questions

I. Resolve the following into partial fractions.

1. $\frac{2x+3}{(x+1)(x-3)}$

Sol. Let $\frac{2x+3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$

Taking lcm and equating both sides

$2x+3 = A(x-3) + B(x+1)$

$x = -1 \Rightarrow 1 = A(-4) \Rightarrow A = -\frac{1}{4}$

$x = 3 \Rightarrow 9 = B(4) \Rightarrow B = \frac{9}{4}$

$\frac{2x+3}{(x+1)(x-3)} = \frac{-1}{4(x+1)} + \frac{9}{4(x-3)}$

2. $\frac{5x+6}{(2+x)(1-x)}$

Sol. Let $\frac{5x+6}{(2+x)(1-x)} = \frac{A}{2+x} + \frac{B}{1-x}$

Multiplying with (2 + x) (1 – x)

5x + 6 = A(1 – x) + B(2 + x)

Put x = –2, –10 + 6 = A(1 + 2) ⇒ A = $-\frac{4}{3}$

Put x = 1, 5 + 6 = B(2 + 1) ⇒ B = $\frac{11}{3}$

$\therefore \frac{5x+6}{(2+x)(1-x)} = -\frac{4}{3(2+x)} + \frac{11}{3(1-x)}$

Short Answer Questions

1. $\frac{3x+7}{x^2-3x+2}$

Sol: We know that

$$\frac{3x+7}{x^2-3x+2}=\frac{3x+7}{(x-2)(x-1)}$$

Let $\frac{3x+7}{x^2-3x+2}=\frac{A}{(x-2)}+\frac{B}{(x-1)}$

$\Rightarrow A(x-1)+B(x-2)=3x+7 \quad \ldots (1)$

Substituting x = 2 in (1)

We get A = 13

Substituting x = 1 in (1)

We get –B = 10 i.e., B = –10

$$\therefore \frac{3x+7}{x^2-3x+2}=\frac{13}{x-2}-\frac{10}{x-1}$$

2. $\frac{x+4}{(x^2-4)(x+1)}$

Sol. $\frac{x+4}{(x^2-4)(x+1)}=\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x-2}$

Taking LCM and equating both sides

$$x+4=A(x^2-4)+B(x+1)(x-2)+C(x+1)(x+2)$$

$x=-1\Rightarrow 3=A(1-4)=-3A\Rightarrow A=-1$

$x=-2\Rightarrow 2=B(-2+1)(-2-2)=4B$

$\Rightarrow B=\frac{2}{4}=\frac{1}{2}$

$x=2\Rightarrow 6=C(2+1)(2+2)=12C\Rightarrow C=\frac{1}{2}$

$$\therefore \frac{x+4}{(x^2-4)(x+1)}=-\frac{1}{x+1}+\frac{1}{2(x+2)}+\frac{1}{2(x-2)}$$

3. $\dfrac{2x^2+2x+1}{x^3+x^2}$

Sol. Let $\dfrac{2x^2+2x+1}{x^3+x^2}=\dfrac{2x^2+2x+1}{x^2(x+1)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x+1}$

Multiplying with $x^2(x + 1)$

$2x^2 + 2x + 1 = A\,x(x + 1) + B(x + 1) + Cx^2$

Put $x = 0$, $1 = B$

Put $x = -1$, $2 - 2 + 1 = C(1) \Rightarrow C = 1$

Equating the coefficients of x^2,

$2 = A + C \Rightarrow A = 2 - C = 2 - 1 = 1$

$\therefore \dfrac{2x^2+2x+1}{x^3+x^2}=\dfrac{1}{x}+\dfrac{1}{x^2}+\dfrac{1}{x+1}$

4. $\dfrac{2x+3}{(x-1)^3}$

Sol. Put $x - 1 = y \Rightarrow x = y + 1$

$\Rightarrow \dfrac{2x+3}{(x-1)^3}=\dfrac{2(y+1)+3}{y^3}=\dfrac{2y+5}{y^3}$

$=\dfrac{2}{y^2}+\dfrac{5}{y^3}=\dfrac{2}{(x-1)^2}+\dfrac{5}{(x-1)^3}$

$\therefore \dfrac{2x+3}{(x-1)^3}=\dfrac{2}{(x-1)^2}+\dfrac{5}{(x-1)^3}$

5. $\dfrac{x^2-2x+6}{(x-2)^3}$

Sol: Let x – 2 = y

$$\therefore \frac{x^2-2x+6}{(x-2)^3}=\frac{(y+2)^2-2(y+2)+6}{y^3}$$

$$=\frac{y^2+4y+4-2y-4+6}{y^3}$$

$$=\frac{y^2+2y+6}{y^3}$$

$$=\frac{1}{y}+\frac{2}{y^2}+\frac{6}{y^3}$$

$$=\frac{1}{x-2}+\frac{2}{(x-2)^2}+\frac{6}{(x-2)^3}$$

$$\therefore \frac{x^2-2x+6}{(x-2)^3}=\frac{1}{x-2}+\frac{2}{(x-2)^2}+\frac{6}{(x-2)^3}$$

6. $\dfrac{2x^2+3x+4}{(x-1)(x^2+2)}$

Sol. Let $\dfrac{2x^2+3x+4}{(x-1)(x^2+2)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+2}$

Multiplying with $(x-1)(x^2+2)$

$$2x^2+3x+4=A(x^2+2)+(Bx+C)(x-1)$$

$$x=1\Rightarrow 2+3+4=A(1+2)$$

$$9=3A\Rightarrow A=3$$

Equating the coefficients of x^2

$2 = A + B \Rightarrow B = 2 - A = 2 - 3 = -1$

Equating constants

$4 = 2A - C \Rightarrow C = 2A - 4 = 6 - 4 = 2$

$$\frac{2x^2+3x+4}{(x-1)(x^2+2)}=\frac{3}{x-1}+\frac{-x+2}{x^2+2}$$

7. $\dfrac{3x-1}{(1-x+x^2)(x+2)}$

Sol. Let $\dfrac{3x-1}{(1-x+x^2)(x+2)}=\dfrac{A}{2+x}+\dfrac{Bx+C}{1-x+x^2}$

Multiplying with $(2 + x)(1 – x + x^2)$

$3x – 1 = A(1 – x + x^2) (Bx + C)(2 + x)$

$x = –2 \Rightarrow –7 = A(1+2+4) = 7A \Rightarrow A = –1$

Equating the coefficients of x^2

$0 = A + B \Rightarrow B = –A = 1$

Equating the constants $–1 = A + 2C$

$2C = –1 – A = –1 + 1 = 0 \Rightarrow C = 0$

$$\frac{3x-1}{(1-x+x^2)(x+2)}=-\frac{1}{2+x}+\frac{x}{1-x+x^2}$$

8. $\dfrac{x^2-3}{(x+2)(x^2+1)}$

Sol. Let $\dfrac{x^2-3}{(x+2)(x^2+1)}=\dfrac{A}{x+2}+\dfrac{Bx+C}{x^2+1}$

Multiplying with $(x + 2)(x^2 + 1)$

$x^2-3=A(x^2+1)+(Bx+C)(x+2)$

$x=-2\Rightarrow 4-3=A(4+1)$

$1=5A\Rightarrow A=\dfrac{1}{5}$

Equating the coefficients of x^2

$$1 = A + B \Rightarrow B = 1 - A = 1 - \frac{1}{5} = \frac{4}{5}$$

Equating the constants –3 = A + 2C

$$2C = -3 - A = -3 - \frac{1}{5} = -\frac{16}{5} \Rightarrow C = -\frac{8}{5}$$

$$\frac{x^2 - 3}{(x+2)(x^2+1)} = \frac{1}{5(x+2)} + \frac{4x-8}{5(x^2+1)}$$

9. $\frac{x^2+1}{(x^2+x+1)^2}$

Sol. Let $\frac{x^2+1}{(x^2+x+1)^2} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2}$

Multiplying with $(x^2 + x + 1)^2$

$x^2 + 1 = (Ax + B)(x^2 + x + 1) + (Cx + D)$

Equating the coefficients of x^3, A = 0

Equating the coefficients of x^2,

A + B = 1 $\Rightarrow$ B = 1

Equating the coefficients of x,

A + B + C = 0

$\Rightarrow$ 1 + C = 0 $\Rightarrow$ C = –1

Equating the constant, B + D = 1

$\Rightarrow$ D = 1 – B = 1 – 1 = 0

$\therefore$ Ax + B = 1, Cx + D = –x

$$\therefore\ \frac{x^2+1}{(x^2+x+1)^2}=\frac{1}{x^2+x+1}-\frac{x}{(x^2+x+1)^2}$$

10. $\dfrac{x^3+x^2+1}{(x-1)(x^3-1)}$

Sol. Let $\dfrac{x^3+x^2+1}{(x-1)(x^3-1)}=\dfrac{x^3+x^2+1}{(x-1)(x-1)(x^2+x+1)}$

Let $\dfrac{x^3+x^2+1}{(x-1)(x-1)(x^2+x+1)}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{Cx+D}{x^2+x+1}$ ...(1)

$$=\frac{A(x-1)(x^2+x+1)+B(x^2+x+1)+(Cx+D)(x-1)^2}{(x-1)^2(x^2+x+1)}$$

$\therefore x^3+x^2+1=A(x-1)(x^2+x+1)+B(x^2+x+1)+(Cx+D)(x-1)^2$ ...(2)

Put $x=1$ in (2)

$1+1+1=A(0)+B(1+1+1)+(C(1)+D)(0)$

$\Rightarrow 3B=3\Rightarrow B=1$

Equating the coefficients of x^3 in (2)

$1 = A + C$ …(3)

Equating the coefficients of x^2 in (2)

$1 = A(1 – 1) + B(1) + C(–2) + D(1)$

$\Rightarrow 1 = B – 2C + D$

$\because B=1\Rightarrow 1=1-2C+D\Rightarrow 2C=D$ …(4)

Put x = 0 in (2)

$1 = A(–1)(1) + B(1) + D(–1)^2$

$\Rightarrow –A + B + D = 1 \Rightarrow –A + 1 + D = 1$

$\Rightarrow A = D$ …(5)

From (3), (4) and (5)

$$1 = D + \frac{D}{2} \Rightarrow \frac{3D}{2} = 1 \Rightarrow D = \frac{2}{3}$$

From (5) $A = \frac{2}{3}$, from (4) $C = \frac{D}{2} = \frac{(2/3)}{2} = \frac{1}{3}$

$$\therefore \frac{x^3 + x^2 + 1}{(x-1)(x^3-1)} = \frac{(2/3)}{x-1} + \frac{1}{(x-1)^2} + \frac{\left(\frac{1}{3}x + \frac{2}{3}\right)}{x^2 + x + 1}$$

$$\Rightarrow \frac{x^3 + x^2 + 1}{(x-1)(x^3-1)} = \frac{2}{3(x-1)} + \frac{1}{(x-1)^2} + \frac{x+2}{3(x^2 + x + 1)}$$

11. $\frac{x^2}{(x-1)(x-2)}$

Sol. Let $\frac{x^2}{(x-1)(x-2)} = 1 + \frac{A}{x-1} + \frac{B}{x-2}$

Multiplying with (x – 1)(x – 2)

$$x^2 = (x-1)(x-2) + A(x-2) + B(x-1)$$

Put x = 1, 1 = A(–1) $\Rightarrow$ A = –1

Put x = 2, 4 = B(1) $\Rightarrow$ B = 4

$$\therefore \frac{x^2}{(x-1)(x-2)} = 1 - \frac{1}{x-1} + \frac{4}{x-2}$$

14. $\dfrac{x^3}{(x-1)(x+2)}$

Sol. Let $\dfrac{x^3}{(x-1)(x+2)} = \dfrac{x^3}{x^2+x-2}$

$$= \frac{x(x^2+x-2)-1(x^2+x-2)+3x-2}{x^2+x-2}$$

$$= x-1+\frac{3x-2}{x^2+x-2} = x-1+\frac{3x-2}{(x-1)(x+2)}$$

Let $\dfrac{3x-2}{(x-1)(x+2)} = \dfrac{A}{x-1}+\dfrac{B}{x+2}$

Multiplying with $(x-1)(x+2)$

$3x-2 = A(x+2)+B(x-1)$

Put $x = 1$, $1 = A(3) \Rightarrow A = \dfrac{1}{3}$

Put $x = -2$, $-8 = B(-3) \Rightarrow B = \dfrac{8}{3}$

$$\therefore \frac{x^3}{(x-1)(x+2)} = x-1+\frac{1}{3(x-1)}+\frac{8}{3(x+2)}$$

15. $\dfrac{x^3}{(2x-1)(x-1)^2}$

Sol. Let

$$\frac{x^3}{(2x-1)(x-1)^2} = \frac{1}{2}+\frac{A}{2x-1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

Multiplying with $2(2x-1)(x-1)^2$

$$2x^3 = (2x-1)(x-1)^2+2A(x-1)^2+2B(2x-1)(x-1)+2C(2x-1)$$

Put $x = \frac{1}{2}, 2\left(\frac{1}{8}\right) = 2A\left(\frac{1}{4}\right) \Rightarrow A = \frac{1}{2}$

Put x = 1, 2(1) = 2C(1) $\Rightarrow$ C = 1

Put x = 0,

0 = (–1)(1) + 2A(1) + 2B(–1)(–1) + 2C(–1)

$\Rightarrow$ 2A + 2B – 2C = 1

$\Rightarrow$ 2B = 1 + 2C – 2A = 1 + 2 – 1 = 2 $\Rightarrow$ B = 1

$$\therefore \frac{x^3}{(2x-1)(x-1)^2} = \frac{1}{2} + \frac{1}{2(2x-1)} + \frac{1}{(x-1)} + \frac{1}{(x-1)^2}$$

16. $\frac{x^3}{(x-a)(x-b)(x-c)}$

Sol. Let $\frac{x^3}{(x-a)(x-b)(x-c)} = 1 + \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

Multiplying with (x – a)(x – b)(x – c)

$$x^3 = (x-a)(x-b)(x-c) + A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)$$

Put x = a, a^3 = A(a – b)(a – c)

$$\Rightarrow A = \frac{a^3}{(a-b)(a-c)}$$

Put x = b, b^3 = B(b – a)(b – c)

$$\Rightarrow B = \frac{b^3}{(b-a)(b-c)}$$

Put x = c, c^3 = C(c – a)(c – b)

$$\Rightarrow C = \frac{c^3}{(c-a)(c-b)}$$

$$\frac{x^3}{(x-a)(x-b)(x-c)} = 1 + \frac{a^3}{(a-b)(a-c)(x-a)} + \frac{b^3}{(b-a)(b-c)(x-b)} + \frac{c^3}{(c-a)(c-b)(x-c)}$$

17. Resolve $\frac{3x-18}{x^3(x+3)}$ into partial fractions.

Sol. Let $\frac{3x-18}{x^3(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+3}$

$$\therefore \frac{3x-18}{x^3(x+3)} = \frac{Ax^2(x+3) + Bx(x+3) + C(x+3) + Dx^3}{x^3(x+3)}$$

$\Rightarrow 3x-18 = Ax^2(x+3) + Bx(x+3) + C(x+3) + Dx^3$ ...(1)

Put x = –3 in (1)

$3(-3) - 18 = A(0) + B(0) + C(0) + D(-3)^3$

$\Rightarrow -27D = -27 \Rightarrow D = 1$

Put x = 0 in (1)

$3(0) - 18 = A(0) + B(0) + C(0+3) + D(0)$

$\Rightarrow 3C = -18 \Rightarrow C = -6$

Equating the coefficients of x^3 in (1)

$0 = A + D \Rightarrow A = -D = -1 \Rightarrow A = -1$

Equating the coefficients of x^2 in (1)

$0 = 3A + B \Rightarrow B = -3A = -3(-1) = 3 \Rightarrow B = 3$

$$\frac{3x-18}{x^3(x+3)} = \frac{-1}{x} + \frac{3}{x^2} - \frac{6}{x^3} + \frac{1}{x+3}$$

7. Resolve $\frac{2x^2+1}{x^3-1}$ into partial fractions.

Sol. $\frac{2x^2+1}{x^3-1} = \frac{2x^2+1}{(x-1)(x^2+x+1)}$ ...(1)

$$\frac{2x^2+1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$\Rightarrow \frac{2x^2+1}{(x-1)(x^2+x+1)} = \frac{A(x^2+x+1)+(Bx+C)(x-1)}{(x-1)(x^2+x+1)}$$

$$\therefore 2x^2+1 = A(x^2+x+1)+(Bx+C)(x-1)...(1)$$

Put x = 1 in (1)

2(1) + 1= A(1 + 1 + 1) + (B + C)(0)

$\Rightarrow$ 3A = 3 $\Rightarrow$ A = 1

Put x = 0 in (1)

0 + 1 = A(1) + (0 + C)(0 – 1)

$\Rightarrow$ 1 = A – C $\Rightarrow$ C = 0

Equating the coefficients of x^2 in (1)

2 = A + B $\Rightarrow$ 2 = 1 + B $\Rightarrow$ B = 1

$$\therefore \frac{2x^2+1}{x^3-1} = \frac{1}{x-1} + \frac{(1)(x)+0}{x^2+x+1} = \frac{1}{x-1} + \frac{x}{x^2+x+1}$$

8. Resolve $\frac{x^3+x^2+1}{(x^2+2)(x^2+3)}$ into partial fractions.

Sol.Let $\frac{x^3+x^2+1}{(x^2+2)(x^2+3)}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{x^2+3}$

$=\frac{(Ax+B)(x^2+3)+(Cx+D)(x^2+2)}{(x^2+2)(x^2+3)}$

$\therefore x^3+x^2+1=(Ax+B)(x^2+3)+(Cx+D)(x^2+2)$ ...(1)

$\Rightarrow x^3+x^2+1=(A+C)x^3+(B+D)x^2+(3A+2C)x+(3B+2D)$

Comparing the coefficients of x^3, x^2, x and constant terms

A + C = 1, B + D = 1,

3A + 2C = 0, 3B + 2D = 1

Solve A = –2, C = 3, B = –1, D = 2

$\therefore \frac{x^3+x^2+1}{(x^2+2)(x^2+3)}=\frac{-2x-1}{x^2+2}+\frac{3x+2}{x^2+3}$

$=\frac{3x+2}{x^2+3}-\frac{2x+1}{x^2+2}$

9. Resolve $\frac{3x^3-2x^2-1}{x^4+x^2+1}$ into partial fractions.

Sol. $x^4+x^2+1=x^4+2x^2+1-x^2$

$=(x^2+1)^2-x^2=(x^2+x+1)(x^2-x+1)$

$\frac{3x^3-2x^2-1}{x^4+x^2+1}=\frac{3x^3-2x^2-1}{(x^2+x+1)(x^2-x+1)}$

$=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2-x+1}$

Multiplying with x^4+x^2+1,

$3x^3 - 2x^2 - 1 = (Ax + B)(x^2 - x + 1) + (Cx + D)(x^2 + x + 1)$

Equating the coefficients of like terms, we get

$A + C = 3$ … (1)

$\Rightarrow C = 3 - A$

$-A + B + C + D = -2$ … (2)

$A - B + C + D = 0$ … (3)

$B + D = -1$ … (4) $D = -1 - B$

Substitute (C), (D) in (2)

$-A + B + 3 - A - 1 - B = -2$

$\Rightarrow -2A = -4 \Rightarrow A = 2$

Substitute C, D in (3)

$A - B + 3 - A - 1 - B = 0 \Rightarrow 2 = 2B \Rightarrow B = 1$

$\therefore C = 3 - 2 = 1, D = -1 - 1 = -2$

$Ax + B = 2x + 1, Cx + D = x - 2$

$$\therefore \frac{3x^3 - 2x^2 - 1}{x^4 + x^2 + 1} = \frac{2x+1}{x^2 + x + 1} + \frac{x-2}{x^2 - x + 1}$$

10. Resolve $\frac{x^4 + 24x^2 + 28}{(x^2+1)^3}$ into partial fractions.

Sol. $\frac{x^4 + 24x^2 + 28}{(x^2+1)^3}$

Put $x^2 + 1 = y \Rightarrow x^2 = y - 1$

Now $\frac{x^4 + 24x^2 + 28}{(x^2+1)^3} = \frac{(y-1)^2 + 24(y-1) + 28}{y^3}$

$$= \frac{y^2 - 2y + 1 + 24y - 24 + 28}{y^3} = \frac{y^2 + 22y + 5}{y^3}$$

$$= \frac{y^2}{y^3} + \frac{22y}{y^3} + \frac{5}{y^3} = \frac{1}{y} + \frac{22}{y^2} + \frac{5}{y^3}$$

$$= \frac{1}{(x^2+1)} + \frac{22}{(x^2+1)^2} + \frac{5}{(x^2+1)^3}$$

$$\therefore \frac{x^4 + 24x^2 + 28}{(x^2+1)^3} = \frac{1}{(x^2+1)} + \frac{22}{(x^2+1)^2} + \frac{5}{(x^2+1)^3}$$

11. Resolve $\frac{x^3}{(2x-1)(x+2)(x-3)}$ into partial fractions.

Sol. $\frac{x^3}{(2x-1)(x+2)(x-3)} = \frac{1}{2} + \frac{A}{2x-1} + \frac{B}{x+2} + \frac{C}{x-3}$

Multiplying with 2(2x – 1)(x + 2)(x – 3)

$$2x^3 = (2x-1)(x+2)(x-3) + 2A(x+2)$$
$$(x-3) + 2B(2x-1)(x-3) + 2C(2x-1)(x+2)$$

Put $x = \frac{1}{2}, 2\left(\frac{1}{8}\right) = 2A\left(\frac{5}{2}\right)\cdot\left(-\frac{5}{2}\right) \Rightarrow A = -\frac{1}{50}$

Put $x = -2$, $2(-8) = 2B(-5)(-5) \Rightarrow B = \frac{-8}{25}$

Put $x = 3$, $2(27) = 2C(5)(5) \Rightarrow C = \frac{27}{25}$

$$\therefore \frac{x^3}{(2x-1)(x+2)(x-3)} = \frac{1}{2} - \frac{1}{50(2x-1)} - \frac{8}{25(x+2)} + \frac{27}{25(x-3)}$$

12. Resolve $\frac{x^4}{(x-1)(x-2)}$ into partial fractions.

Sol. $\frac{x^4}{(x-1)(x-2)} = \frac{x^4}{x^2-3x+2}$

$= \frac{x^2(x^2-3x+2)+3x(x^2-3x+2)+7(x^2-3x+2)+15x-14}{x^2-3x+2}$

$= x^2+3x+7+\frac{15x-14}{x^2-3x+2}$

Let $\frac{15x-14}{(x-1)(x-2)} = \frac{A}{x-1}+\frac{B}{x-2}$

Equating the coefficients of (x – 1)(x – 2)

$15x-14 = A(x-2)+B(x-1)$

Put x = 1, 15 – 14 = A(–1) ⇒ A = –1

Put x = 2, 30 – 14 = B(1) ⇒ B = 16

$\therefore \frac{x^4}{(x-1)(x-2)} = x^2+3x+7-\frac{1}{x-1}+\frac{16}{x-2}$

13. Find the coefficient of x^4 in the expansion of $\frac{3x}{(x-2)(x+1)}$.

Sol. $\frac{3x}{(x-2)(x+1)} = \frac{A}{x-2}+\frac{B}{x+1}$

Multiplying with (x – 2)(x + 1)

3x = A(x + 1) + B(x – 2)

Put x = –1, –3 = B(–3) ⇒ B = 1

Put x = 2, 6 = A(3) ⇒ A = 2

$$\therefore \frac{3x}{(x-2)(x+1)} = \frac{2}{x-2} + \frac{1}{x+1}$$

$$= \frac{2}{-2\left(1-\frac{x}{2}\right)} + \frac{1}{1+x} = -\left(1-\frac{x}{2}\right)^{-1} + (1+x)^{-1}$$

$$= -\left[1+\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{16}+...\right]+\left(1-x+x^2-x^3+x^4...\right)$$

$$\therefore \text{Coefficient of } x^4 = -\frac{1}{16}+1=\frac{15}{16}$$

14. Find the coefficient of x^n in the expansion of $\frac{x}{(x-1)^2(x-2)}$.

Sol. $\frac{x}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$

Multiplying with $(x – 1)^2 (x – 2)$

$x = A(x – 1)(x – 2) + B(x – 2) + C(x – 1)^2$

Put $x = 1$, $1 = B(–1) \Rightarrow B = –1$

Put $x = 2$, $2 = C(1) \Rightarrow C = 2$

Put $x = 0$, $0 = 2A – 2B + C \Rightarrow 2A = 2B – C$

$= –2 – 2 = –4 \Rightarrow A = –2$

$$\therefore \frac{x}{(x-1)^2(x-2)} = \frac{-2}{x-1} - \frac{1}{(x-1)^2} + \frac{2}{x-2}$$

$$= \frac{2}{1-x} - \frac{1}{(1-x)^2} + \frac{2}{-2\left(1-\frac{x}{2}\right)}$$

$$= 2(1-x)^{-1} - (1-x)^{-2} - \left(1-\frac{x}{2}\right)^{-1}$$

$$= 2[1+x+x^2+...+x^n+...]-[1+2x+3x^2+...+(n+1)x^n+...]-\left[1+\frac{x^2}{2}+\frac{x^2}{4}+...\frac{x^n}{2^n}+...\right]$$

$\therefore$ Coefficient of $x^n = 2(1)-(n+1)-\left(\frac{1}{2^n}\right)$

$$=2-n-1-\frac{1}{2^n}=1-n-\frac{1}{2^n}.$$

15.Resolve $\frac{x+3}{(1-x)^2(1+x^2)}$ in to partial fractions.

Sol: Let $\frac{x+3}{(1-x)^2(1+x^2)}$

$$=\frac{A}{(1-x)}+\frac{B}{(1-x)^2}+\frac{Cx+D}{(1+x^2)}$$

$$\Rightarrow x+3=A(1-x)(1+x^2)+B(1+x^2)+(Cx+D)(1-x)^2$$

Comparing the coefficients of like power of x, we get

$A + B + D = 3$ … (1)

$-A + C - 2D = 1$ … (2)

$A + B - 2C + D = 0$ … (3)

$-A + C = 0$ … (4)

Solving above equations, we get

$$A=\frac{3}{2}, B=2, C=\frac{3}{2}, D=-\frac{1}{2}$$

$$\therefore \frac{x+3}{(1-x)^2(1+x^2)}=\frac{3}{2(1-x)}+\frac{2}{(1-x)^2}+\frac{3x-1}{2(1+x^2)}$$

16. Resolve $\frac{x^2+5x+7}{(x-3)^3}$ into partial fractions.

Sol.Let $x - 3 = y \Rightarrow x = y + 3$

$$\frac{x^2+5x+7}{(x-3)^3}=\frac{(y+3)^2+5(y+3)+7}{y^3}$$

$$=\frac{y^2+6y+9+5y+15+7}{y^3}$$

$$= \frac{y^2 + 11y + 31}{y^3} = \frac{1}{y} + \frac{11}{y^2} + \frac{31}{y^3}$$

$$\therefore \frac{x^2 + 5x + 7}{(x-3)^3} = \frac{1}{x-3} + \frac{11}{(x-3)^2} + \frac{31}{(x-3)^3}$$

17. Write $\frac{x^2 + 13x + 15}{(2x+3)(x+3)^2}$ as a sum of partial fractions.

Sol. Let $\frac{x^2 + 13x + 15}{(2x+3)(x+3)^2} = \frac{A}{2x+3} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$

Multiplying $(2x + 3)(x + 3)^2$

$$x^2 + 13x + 15 = A(x+3)^2 + B(x+3)(2x+3) + C(2x+3)$$

Put $x = -\frac{3}{2}, \frac{9}{4} - \frac{39}{2} + 15 = A\left(\frac{9}{4}\right)$

$$\Rightarrow \frac{9A}{4} = \frac{9 - 78 + 60}{4} = -\frac{9}{4} \Rightarrow A = -1$$

Put $x = -3$, $9 - 39 + 15 = C(-3)$

$\Rightarrow -3C = -15 \Rightarrow C = 5$

Equating the coefficients of x^2,

$A + 2B = 1 \Rightarrow 2B = 1 - A = 1 + 1 = 2 \Rightarrow B = 1$

$$\therefore \frac{x^2 + 13x + 15}{(2x+3)(x+3)^2} = \frac{-1}{2x+3} + \frac{1}{x+3} + \frac{5}{(x+3)^2}$$

18. Resolve $\frac{1}{(x-1)^2(x-2)}$ into partial fractions.

Sol.Let $\frac{1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$

$$\Rightarrow \frac{1}{(x-1)^2(x-2)}$$

$$= \frac{A(x-1)(x-2) + B(x-2) + C(x-1)^2}{(x-1)^2(x-2)}$$

$$\Rightarrow 1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2 \ldots(1)$$

Put x = 1 in (1)

1 = A(0) + B(1 – 2) + C(0)

$\Rightarrow$ –B = 1 $\Rightarrow$ B = –1

Put x = 2 in (1)

1 = A(0) + B(0) + C$(2 – 1)^2$ $\Rightarrow$ C = 1

Equating the coefficients of x^2 in (1)

0 = A + C $\Rightarrow$ A = –C = –1 $\Rightarrow$ A = –1

$$\therefore \frac{1}{(x-1)^2(x-2)} = \frac{-1}{x-1} - \frac{1}{(x-1)^2} + \frac{1}{x-2}$$

III.

1. $\dfrac{x^2 - x + 1}{(x+1)(x-1)^2}$

Sol. Let $\dfrac{x^2 - x + 1}{(x+1)(x-1)^2} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$

Multiplying with $(x + 1)(x – 1)^2$

$x^2 – x + 1 =$

$A(x – 1)^2 + B(x + 1)(x – 1) + C(x + 1)$

Put $x = –1$, $1 + 1 + 1 = A(4) \Rightarrow A = \dfrac{3}{4}$

Put $x = 1$, $1 – 1 + 1 = C(2) \Rightarrow C = \dfrac{1}{2}$

Equating the coefficients of x^2

$A + B = 1 \Rightarrow B = 1 – A = 1 - \dfrac{3}{4} = \dfrac{1}{4}$

$\therefore \dfrac{x^2 - x + 1}{(x+1)(x-1)^2} = \dfrac{3}{4(x+1)} + \dfrac{1}{4(x-1)} + \dfrac{1}{2(x-1)^2}$

2. $\dfrac{9}{(x-1)(x+2)^2}$

Sol. Let $\dfrac{9}{(x-1)(x+2)^2} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$

Multiplying with $(x – 1)(x + 2)^2$

$9 = A(x + 2)^2 + B(x – 1)(x + 2) + C(x – 1)$

$x = 1 \Rightarrow 9 = 9A \Rightarrow A = 1$

$x = –2 \Rightarrow 9 = –3C \Rightarrow C = –3$

Equating the coefficients of x^2

$A + B = 0 \Rightarrow B = -A = -1$

$$\therefore \frac{9}{(x-1)(x+2)^2} = \frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}$$

3. $\dfrac{x-1}{(x+1)(x-2)^2}$

Sol. Let $\dfrac{x-1}{(x+1)(x-2)^2} = \dfrac{A}{x+1} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$

$\Rightarrow x - 1 =$

$A(x-2)^2 + B(x-2)(x+1) + C(x+1) \ldots(1)$

Put $x = 2$ in (1)

$2 - 1 = A(0) + B(0) + C(2 + 1)$

$\Rightarrow 1 = 3C \Rightarrow C = \dfrac{1}{3}$

Put $x = -1$ in (1)

$-1 -1 = A(-1-2)^2 + B(0) + C(0)$

$\Rightarrow 9A = -2 \Rightarrow A = \dfrac{-2}{9}$

Equating the coefficients of x^2 in (1)

$0 = A + B \Rightarrow B = -A = \dfrac{2}{9}$

$$\therefore \frac{x-1}{(x+1)(x-2)^2} = \frac{\frac{-2}{9}}{x+1} + \frac{\frac{2}{9}}{x-2} + \frac{\frac{1}{3}}{(x-2)^2}$$

$$\frac{x-1}{(x+1)(x-2)^2} = \frac{-2}{9(x+1)} + \frac{2}{9(x-2)} + \frac{1}{3(x-2)^2}$$

4. $\frac{1}{(1-2x)^2(1-3x)}$

Sol. Let

$$\frac{1}{(1-2x)^2(1-3x)}=\frac{A}{1-3x}+\frac{B}{1-2x}+\frac{C}{(1-2x)^2}$$

Multiplying with $(1-2x)^2(1-3x)$

$$1=A(1-2x)^2+B(1-3x)(1-2x)+C(1-3x)$$

$$x=\frac{1}{3}\Rightarrow 1=A\left(1-\frac{2}{3}\right)^2=\frac{A}{9}\Rightarrow A=9$$

$$x=\frac{1}{2}\Rightarrow 1=C\left(1-\frac{3}{2}\right)=-\frac{C}{2}\Rightarrow C=-2$$

Equating the coefficients of x^2

$0 = 4A + 6B$

$6B = -4A = -36$

$B = -6$

$$\frac{1}{(1-2x)^2(1-3x)}=\frac{9}{1-3x}-\frac{6}{1-2x}-\frac{2}{(1-2x)^2}$$

5. $\frac{1}{x^3(x+a)}$

Sol. Let $\frac{1}{x^3(x+a)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x+a}$

$$=\frac{A\cdot x^2(x+a)+B(a)(x+a)+C(x+a)+Dx^3}{x^3(x+a)}$$

$$\therefore 1=A(x^2)(x+a)+Bx(x+a)+C(x+a)+Dx^3\ldots(1)$$

Put x = 0 in (1)

$1 = A(0) + B(0) + C(0 + a) + D(0)$

$\Rightarrow 1 = C(a) \Rightarrow C = \frac{1}{a}$

Put x = –a in (1)

$1 = A(0) + B(0) + C(0) + D(-a)^3$

$\Rightarrow 1 = -Da^2 \Rightarrow D = -\frac{1}{a^3}$

Equating the coefficients of x^3 in (1)

$0 = A + D$

$\Rightarrow A = -D = \frac{1}{a^3}, A = \frac{1}{a^3}$

Equating the coefficients of x^2 in (1)

$0 = Aa + B$

$\Rightarrow B = -aA = -a\left(\frac{1}{a^3}\right) = -\frac{1}{a^2}$

$\therefore B = -\frac{1}{a^2}$

$$\frac{1}{x^3(x+a)} = \frac{\left(\frac{1}{a^3}\right)}{x} + \frac{\left(-\frac{1}{a^2}\right)}{x^2} + \frac{\left(\frac{1}{a}\right)}{x^3} + \frac{\left(-\frac{1}{a^3}\right)}{x+a}$$

$$\Rightarrow \frac{1}{x^3(x+a)} = \frac{1}{a^3x} - \frac{1}{a^2x^2} + \frac{1}{ax^3} - \frac{1}{a^3(x+a)}$$

6. $\dfrac{3x^3 - 8x^2 + 10}{(x-1)^4}$

Sol: Let x – 1 = y

$$\therefore \frac{3x^3 - 8x^2 + 10}{(x-1)^4}$$

$$= \frac{3(y+1)^3 - 8(y+1)^2 + 10}{y^4}$$

$$= \frac{3(y^3 + 3y^2 + 3y + 1) - 8(y^2 + 2y + 1) + 10}{y^4}$$

$$= \frac{3y^3 + y^2 - 7y + 5}{y^4}$$

$$= \frac{3}{y} + \frac{1}{y^2} - \frac{7}{y^3} + \frac{5}{y^4}$$

$$= \frac{3}{x-1} + \frac{1}{(x-1)^2} - \frac{7}{(x-1)^3} + \frac{5}{(x-1)^4}$$

$$\therefore \frac{3x^3 - 8x^2 + 10}{(x-1)^4}$$

$$= \frac{3}{x-1} + \frac{1}{(x-1)^2} - \frac{7}{(x-1)^3} + \frac{5}{(x-1)^4}$$

7. Resolve $\dfrac{3x-18}{x^3(x+3)}$ **into partial fractions.**

Sol. Let $\dfrac{3x-18}{x^3(x+3)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{x+3}$

$$\therefore \frac{3x-18}{x^3(x+3)} = \frac{Ax^2(x+3) + Bx(x+3) + C(x+3) + Dx^3}{x^3(x+3)}$$

$$\Rightarrow 3x - 18 = Ax^2(x+3) + Bx(x+3) + C(x+3) + Dx^3 \quad ...(1)$$

Put x = –3 in (1)

$3(–3) – 18 = A(0) + B(0) + C(0) + D(–3)^3$

$\Rightarrow -27D = -27 \Rightarrow D = 1$

Put x = 0 in (1)

$3(0) - 18 = A(0) + B(0) + C(0 + 3) + D(0)$

$\Rightarrow 3C = -18 \Rightarrow C = -6$

Equating the coefficients of x^3 in (1)

$0 = A + D \Rightarrow A = -D = -1 \Rightarrow A = -1$

Equating the coefficients of x^2 in (1)

$0 = 3A + B \Rightarrow B = -3A = -3(-1) = 3 \Rightarrow B = 3$

$$\frac{3x-18}{x^3(x+3)} = \frac{-1}{x} + \frac{3}{x^2} - \frac{6}{x^3} + \frac{1}{x+3}$$

8. Resolve $\frac{2x^2+1}{x^3-1}$ into partial fractions.

Sol. $\frac{2x^2+1}{x^3-1} = \frac{2x^2+1}{(x-1)(x^2+x+1)}$ ...(1)

$$\frac{2x^2+1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$\Rightarrow \frac{2x^2+1}{(x-1)(x^2+x+1)} = \frac{A(x^2+x+1)+(Bx+C)(x-1)}{(x-1)(x^2+x+1)}$$

$\therefore 2x^2+1 = A(x^2+x+1)+(Bx+C)(x-1)$...(1)

Put x = 1 in (1)

$2(1) + 1 = A(1 + 1 + 1) + (B + C)(0)$

$\Rightarrow 3A = 3 \Rightarrow A = 1$

Put x = 0 in (1)

$0 + 1 = A(1) + (0 + C)(0 – 1)$

$\Rightarrow 1 = A – C \Rightarrow C = 0$

Equating the coefficients of x^2 in (1)

$2 = A + B \Rightarrow 2 = 1 + B \Rightarrow B = 1$

$$\therefore \frac{2x^2+1}{x^3-1} = \frac{1}{x-1} + \frac{(1)(x)+0}{x^2+x+1} = \frac{1}{x-1} + \frac{x}{x^2+x+1}$$

9. Resolve $\frac{x^3+x^2+1}{(x^2+2)(x^2+3)}$ into partial fractions.

Sol. Let $\frac{x^3+x^2+1}{(x^2+2)(x^2+3)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+3}$

$$= \frac{(Ax+B)(x^2+3)+(Cx+D)(x^2+2)}{(x^2+2)(x^2+3)}$$

$\therefore x^3+x^2+1 =$

$(Ax+B)(x^2+3)+(Cx+D)(x^2+2)$ ...(1)

$\Rightarrow x^3+x^2+1 = (A+C)x^3+(B+D)x^2+(3A+2C)x+(3B+2D)$

Comparing the coefficients of x^3, x^2, x and constant terms

$A + C = 1$, $B + D = 1$,

$3A + 2C = 0$, $3B + 2D = 1$

Solve $A = –2$, $C = 3$, $B = –1$, $D = 2$

$$\therefore \frac{x^3+x^2+1}{(x^2+2)(x^2+3)} = \frac{-2x-1}{x^2+2} + \frac{3x+2}{x^2+3}$$

$$= \frac{3x+2}{x^2+3} - \frac{2x+1}{x^2+2}$$

10. Resolve $\frac{3x^3-2x^2-1}{x^4+x^2+1}$ into partial fractions.

Sol. $x^4+x^2+1=x^4+2x^2+1-x^2$

$=(x^2+1)^2-x^2=(x^2+x+1)(x^2-x+1)$

$$\frac{3x^3-2x^2-1}{x^4+x^2+1}=\frac{3x^3-2x^2-1}{(x^2+x+1)(x^2-x+1)}$$

$$=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2-x+1}$$

Multiplying with x^4+x^2+1,

$3x^3-2x^2-1=$

$(Ax+B)(x^2-x+1)+(Cx+D)(x^2+x+1)$

Equating the coefficients of like terms, we get

$A+C=3$ … (1)

$\Rightarrow C=3-A$

$-A+B+C+D=-2$ … (2)

$A-B+C+D=0$ … (3)

$B+D=-1$ … (4) $D=-1-B$

Substitute (C), (D) in (2)

$-A+B+3-A-1-B=-2$

$\Rightarrow -2A=-4\Rightarrow A=2$

Substitute C, D in (3)

$A-B+3-A-1-B=0\Rightarrow 2=2B\Rightarrow B=1$

$\therefore C=3-2=1,\ D=-1-1=-2$

$Ax+B=2x+1,\ Cx+D=x-2$

$$\therefore \frac{3x^3-2x^2-1}{x^4+x^2+1}=\frac{2x+1}{x^2+x+1}+\frac{x-2}{x^2-x+1}$$

11. Resolve $\frac{x^3}{(2x-1)(x+2)(x-3)}$ into partial fractions.

Sol. $\frac{x^3}{(2x-1)(x+2)(x-3)}=\frac{1}{2}+\frac{A}{2x-1}+\frac{B}{x+2}+\frac{C}{x-3}$

Multiplying with 2(2x – 1)(x + 2)(x – 3)

$$2x^3=(2x-1)(x+2)(x-3)+2A(x+2)$$
$$(x-3)+2B(2x-1)(x-3)+2C(2x-1)(x+2)$$

Put $x=\frac{1}{2}, 2\left(\frac{1}{8}\right)=2A\left(\frac{5}{2}\right)\cdot\left(-\frac{5}{2}\right)\Rightarrow A=-\frac{1}{50}$

Put x = –2, 2(–8) = 2B(–5)(–5) $\Rightarrow$ B = $\frac{-8}{25}$

Put x = 3, 2(27) = 2C(5)(5) $\Rightarrow$ C = $\frac{27}{25}$

$$\therefore \frac{x^3}{(2x-1)(x+2)(x-3)}=\frac{1}{2}-\frac{1}{50(2x-1)}-\frac{8}{25(x+2)}+\frac{27}{25(x-3)}$$

12. Resolve $\frac{x^4}{(x-1)(x-2)}$ into partial fractions.

Sol. $\frac{x^4}{(x-1)(x-2)}=\frac{x^4}{x^2-3x+2}$

$$=\frac{x^2(x^2-3x+2)+3x(x^2-3x+2)+7(x^2-3x+2)+15x-14}{x^2-3x+2}$$

$$=x^2+3x+7+\frac{15x-14}{x^2-3x+2}$$

Let $\frac{15x-14}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}$

Equating the coefficients of (x – 1)(x – 2)

$15x-14=A(x-2)+B(x-1)$

Put x = 1, 15 – 14 = A(–1) ⇒ A = –1

Put x = 2, 30 – 14 = B(1) ⇒ B = 16

$\therefore \frac{x^4}{(x-1)(x-2)}=x^2+3x+7-\frac{1}{x-1}+\frac{16}{x-2}$

13. Find the coefficient of x^4 in the expansion of $\frac{3x}{(x-2)(x+1)}$.

Sol. $\frac{3x}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}$

Multiplying with (x – 2)(x + 1)

3x = A(x + 1) + B(x – 2)

Put x = –1, –3 = B(–3) ⇒ B = 1

Put x = 2, 6 = A(3) ⇒ A = 2

$\therefore \frac{3x}{(x-2)(x+1)}=\frac{2}{x-2}+\frac{1}{x+1}$

$=\frac{2}{-2\left(1-\frac{x}{2}\right)}+\frac{1}{1+x}=-\left(1-\frac{x}{2}\right)^{-1}+(1+x)^{-1}$

$=-\left[1+\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{16}+...\right]+\left(1-x+x^2-x^3+x^4...\right)$

$\therefore$ Coefficient of $x^4=-\frac{1}{16}+1=\frac{15}{16}$

14. Find the coefficient of x^n in the expansion of $\frac{x}{(x-1)^2(x-2)}$.

Sol. $\frac{x}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$

Multiplying with $(x-1)^2(x-2)$

$x = A(x-1)(x-2) + B(x-2) + C(x-1)^2$

Put $x = 1$, $1 = B(-1) \Rightarrow B = -1$

Put $x = 2$, $2 = C(1) \Rightarrow C = 2$

Put $x = 0$, $0 = 2A - 2B + C \Rightarrow 2A = 2B - C$

$= -2 - 2 = -4 \Rightarrow A = -2$

$\therefore \frac{x}{(x-1)^2(x-2)} = \frac{-2}{x-1} - \frac{1}{(x-1)^2} + \frac{2}{x-2}$

$= \frac{2}{1-x} - \frac{1}{(1-x)^2} + \frac{2}{-2\left(1-\frac{x}{2}\right)}$

$= 2(1-x)^{-1} - (1-x)^{-2} - \left(1-\frac{x}{2}\right)^{-1}$

$= 2[1+x+x^2+...+x^n+...] - [1+2x+3x^2+...+(n+1)x^n+...] - \left[1+\frac{x^2}{2}+\frac{x^2}{4}+...\frac{x^n}{2^n}+...\right]$

$\therefore$ Coefficient of x^n = $2(1)-(n+1)-\left(\frac{1}{2^n}\right)$

$= 2-n-1-\frac{1}{2^n} = 1-n-\frac{1}{2^n}$.

15. Find the coefficients of x^3 in the expansion of $\frac{5x+6}{(2+x)(1-x)}$.

Sol. Let $\frac{5x+6}{(2+x)(1-x)}=\frac{A}{2+x}+\frac{B}{1-x}$

Multiplying with (2 + x)(1 – x)

5x + 6 = A(1 – x) + B(2 + x)

$x=1\Rightarrow 11=B(2+1)=3B\Rightarrow B=\frac{11}{3}$

$x=-2\Rightarrow -4=A(1+2)=3A\Rightarrow A=\frac{-4}{3}$

$$\frac{5x+6}{(2+x)(1-x)}=\frac{-4}{3(2+x)}+\frac{11}{3(1-x)}$$

$$=\frac{-4}{3\cdot 2\left(1+\frac{x}{2}\right)}+\frac{11}{3(1-x)}$$

$$=-\frac{2}{3}\left(1+\frac{x}{2}\right)^{-1}+\frac{11}{3}(1-x)^{-1}$$

$$=-\frac{2}{3}\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+...\right)+\frac{11}{3}\left(1+x+x^2+x^3...\right)$$

$\therefore$ Coefficient of $x^3=-\frac{2}{3}\left(-\frac{1}{8}\right)+\frac{11}{3}(1)$

$$=\frac{2+88}{24}=\frac{90}{24}=\frac{15}{4}$$

16. What is the coefficient of x^4 in the expansion of $\frac{3x^2+2x}{(x^2+2)(x-3)}$.

Sol. Let $\frac{3x^2+2x}{(x^2+2)(x-3)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+2}$

Multiplying with $(x^2 + 2)(x - 3)$

$3x^2+2x = A(x^2+2)+(Bx+C)(x-3)$

$x = 3 \Rightarrow 27+6 = A(9+2)$

$33 = 11A \Rightarrow A = 3$

Equating the coefficients of x^2

$3 = A + B \Rightarrow B = 3 - A = 3 - 3 = 0$

Equating the constants,

$2A - 3C = 0 \Rightarrow 3C = 2A = 6 \Rightarrow C = 2$

$$\frac{3x^2+2x}{(x^2+2)(x-3)} = \frac{3}{x-3} + \frac{2}{x^2+2}$$

$$= \frac{3}{-3\left(1-\frac{x}{3}\right)} + \frac{2}{2\left(1+\frac{x^2}{2}\right)}$$

$$= -\left(1-\frac{x}{3}\right)^{-1} + \left(1+\frac{x^2}{2}\right)^{-1}$$

$$= -\left(1+\frac{x}{3}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{81}+\ldots\right) + \left(1-\frac{x^2}{2}+\frac{x^4}{4}-\ldots\right)$$

$\therefore$ Coefficients of $x^4 = -\frac{1}{81}+\frac{1}{4}$

$$= \frac{-4+81}{324} = \frac{77}{324}$$

17. Find the coefficient of x^n in the expansion of $\frac{x-4}{x^2-5x+6}$.

Sol. Let $\frac{x-4}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}$

Multiplying with (x – 2)(x – 3)

$$x-4=A(x-3)+B(x-2)$$
$$x=2\Rightarrow -2=A(2-3)=-A\Rightarrow A=2$$
$$x=3\Rightarrow -1=B(3-2)=B\Rightarrow B=-1$$

$$\frac{x-4}{x^2-5x+6}=\frac{2}{x-2}-\frac{1}{x-3}$$
$$=\frac{2}{-2\left(1-\frac{x}{2}\right)}+\frac{1}{3\left(1-\frac{x}{3}\right)}$$
$$=-\left(1-\frac{x}{2}\right)^{-1}+\frac{1}{3}\left(1-\frac{x}{3}\right)^{-1}$$
$$=-\left(1+\frac{x}{2}+\frac{x^2}{4}+\ldots+\frac{x^n}{2^n}+\ldots\right)+\frac{1}{3}\left(1+\frac{x}{3}+\frac{x^2}{9}+\ldots+\frac{x^n}{3^n}+\ldots\right)$$

Coefficient of $x^n=\frac{1}{3^{n+1}}-\frac{1}{2^n}$

18. Find the coefficient of x^n in the power series expansion of $\frac{3x}{(x-1)(x-2)^2}$.

Sol: Let

$$\frac{3x}{(x-1)(x-2)^2}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$$

$\Rightarrow A(x-2)^2+B(x-1)(x-2)+C(x-1)=3x$ Substituting x = 1 in (1), we get A = 3

Substituting x = 2 in (2), we get C = 6

Equating coefficient of x^2 in (1) we get

A + B = 0 ⇒ B = –A ⇒ B = –3

$$\therefore \frac{3x}{(x-1)(x-2)^2} = \frac{3}{x-1} - \frac{3}{x-2} + \frac{6}{(x-2)^2}$$

$$= -3(1-x)^{-1} + \frac{3}{2}\left(1-\frac{x}{2}\right)^{-1} + \frac{3}{2}\left(1-\frac{x}{2}\right)^{-2}$$

Now

$$(1-x)^{-1} = 1 + x + x^2 + \ldots + x^n + \ldots \text{if } |x| < 1$$

$$\left(1-\frac{x}{2}\right)^{-1} = 1 + \frac{x}{2} + \frac{x^2}{4} + \ldots + \frac{x^n}{2^n} + \ldots$$

$$\text{if } \left|\frac{x}{2}\right| < 1$$

i.e. |x| < 2

$$\left(1-\frac{x}{2}\right)^{-2} = 1 + 2\left(\frac{x}{2}\right) + 3\left(\frac{x}{2}\right)^2 + 4\left(\frac{x}{2}\right)^3$$

i.e., |x| < 2.

$$+ \ldots + (n+1)\left(\frac{x}{2}\right)^n + \ldots, \text{if } \left|\frac{x}{2}\right| < 1$$

The above expansion are valid when |x| < 1.

$$\therefore \frac{3x}{(x-1)(x-2)^2} = -3\left(1 + x + x^2 + \ldots + x^n + \ldots\right) + \left(1 + \frac{x}{2} + \frac{x^2}{4} + \ldots + \frac{x^n}{2^n} + \ldots\right)$$

$$+ \frac{3}{2}\left(1 + 2\left(\frac{x}{2}\right) + 3\left(\frac{x}{2}\right)^2 + \ldots + (n+1)\left(\frac{x}{2}\right)^n + \ldots\right)$$

Coefficient of x^n in this expansion is

$$-3 + \frac{3}{2}\left(\frac{1}{2^n}\right) + \frac{3}{2}\frac{(n+1)}{2^n}$$

$$= -3 + \frac{3}{2^{n+1}} + \frac{3(n+1)}{2^{n+1}}.$$